

Introduction to the μ -calculus

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Is Hennessy-Milner logic expressive enough?

Is Hennessy-Milner logic expressive enough?

- It cannot detect deadlock in an arbitrary process
- or general *safety*: all reachable states verify ϕ
- or general *liveness*: there is a reachable states which verifies ϕ
- ...

... essentially because

formulas in cannot see deeper than their modal depth

Is Hennessy-Milner logic expressive enough?

Example

$\phi =$ a taxi eventually returns to its Central

$$\phi = \langle \text{reg} \rangle \text{true} \vee \langle - \rangle \langle \text{reg} \rangle \text{true} \vee \langle - \rangle \langle - \rangle \langle \text{reg} \rangle \text{true} \vee \langle - \rangle \langle - \rangle \langle - \rangle \langle \text{reg} \rangle \text{true} \vee \dots$$

Revisiting Hennessy-Milner logic

Adding regular expressions

ie, with regular expressions within modalities

$$\rho ::= \epsilon \mid \alpha \mid \rho.\rho \mid \rho + \rho \mid \rho^* \mid \rho^+$$

where

- α is an **action formula** and ϵ is the **empty word**
- **concatenation** $\rho.\rho$, **choice** $\rho + \rho$ and **closures** ρ^* and ρ^+

Laws

$$\langle \rho_1 + \rho_2 \rangle \phi = \langle \rho_1 \rangle \phi \vee \langle \rho_2 \rangle \phi$$

$$[\rho_1 + \rho_2] \phi = [\rho_1] \phi \wedge [\rho_2] \phi$$

$$\langle \rho_1.\rho_2 \rangle \phi = \langle \rho_1 \rangle \langle \rho_2 \rangle \phi$$

$$[\rho_1.\rho_2] \phi = [\rho_1][\rho_2] \phi$$

Revisiting Hennessy-Milner logic

Examples of properties

- $\langle \epsilon \rangle \phi = [\epsilon] \phi = \phi$
- $\langle a.a.b \rangle \phi = \langle a \rangle \langle a \rangle \langle b \rangle \phi$
- $\langle a.b + g.d \rangle \phi$

Safety

- $[-^*] \phi$
- it is impossible to do two consecutive enter actions without a leave action in between:
 $[-^*.enter. - leave^*.enter] false$
- absence of **deadlock**:
 $[-^*] \langle - \rangle true$

Revisiting Hennessy-Milner logic

Examples of properties

Liveness

- $\langle -^* \rangle \phi$
- after sending a message, it can eventually be received:
 $[send] \langle -^*.receive \rangle true$
- after a send a receive is possible as long as an exception does not happen:
 $[send. - excp^*] \langle -^*.receive \rangle true$

The modal μ -calculus

- modalities with regular expressions are not enough in general
- ... but correspond to a subset of the modal μ -calculus [Kozen83]

Add explicit *minimal/maximal fixed point operators* to Hennessy-Milner logic

$\phi ::= X \mid true \mid false \mid \neg\phi \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \rightarrow \phi \mid \langle a \rangle \phi \mid [a] \phi \mid \mu X. \phi \mid \nu X. \phi$

The modal μ -calculus

The modal μ -calculus (intuition)

- $\mu X . \phi$ is valid for all those states in the *smallest* set X that satisfies the equation $X = \phi$ (finite paths, **liveness**)
- $\nu X . \phi$ is valid for the states in the *largest* set X that satisfies the equation $X = \phi$ (infinite paths, **safety**)

Warning

In order to be sure that a fixed point exists, X must occur positively in the formula, ie **preceded by an even number of negations**.

Temporal properties as limits

Example

$$A \triangleq \sum_{i \geq 0} A_i \quad \text{with} \quad A_0 \triangleq \mathbf{0} \text{ e } A_{i+1} \triangleq a.A_i$$

$$A' \triangleq A + D \quad \text{with} \quad D \triangleq a.D$$

- $A \approx A'$
- but there is no modal formula to distinguish A from A'
- notice $A' \models \langle a \rangle^{i+1} \text{true}$ which A_i fails
- a distinguishing formula would require **infinite** conjunction
- what we want to express is the possibility of doing a in **the long run**

Temporal properties as limits

idea: introduce recursion in formulas

$$X \triangleq \langle a \rangle X$$

meaning?

- the **recursive** formula is interpreted as a *fixed point* of function

$$|\langle a \rangle|$$

in $\mathcal{P}\mathbb{P}$

- i.e., the *solutions*, $S \subseteq \mathbb{P}$ such that of

$$S = |\langle a \rangle|(S)$$

- how do we solve this equation?

Solving equations ...

over natural numbers

$$x = 3x \quad \text{one solution } (x = 0)$$

$$x = 1 + x \quad \text{no solutions}$$

$$x = 1x \quad \text{many solutions (every natural } x)$$

over sets of integers

$$x = \{22\} \cap x \quad \text{one solution } (x = \{22\})$$

$$x = \mathbb{N} \setminus x \quad \text{no solutions}$$

$$x = \{22\} \cup x \quad \text{many solutions (every } x \text{ st } \{22\} \subseteq x)$$

Solving equations ...

In general, for a *monotonic* function f , i.e.

$$X \subseteq Y \Rightarrow f X \subseteq f Y$$

Knaster-Tarski Theorem [1928]

A monotonic function f in a complete lattice has a

- *unique maximal fixed point*:

$$\nu_f = \bigcup \{X \in \mathcal{P}\mathbb{P} \mid X \subseteq f X\}$$

- *unique minimal fixed point*:

$$\mu_f = \bigcap \{X \in \mathcal{P}\mathbb{P} \mid f X \subseteq X\}$$

- moreover the space of its solutions forms a complete lattice

Back to the example ...

$S \in \mathcal{P}\mathbb{P}$ is a *pre-fixed point* of $|\langle a \rangle|$
iff

$$|\langle a \rangle|(S) \subseteq S$$

Recalling,

$$|\langle a \rangle|(S) = \{E \in \mathbb{P} \mid \exists E' \in S . E \xrightarrow{a} E'\}$$

the set of sets of processes we are interested in is

$$\begin{aligned} Pre &= \{S \subseteq \mathbb{P} \mid \{E \in \mathbb{P} \mid \exists E' \in S . E \xrightarrow{a} E'\} \subseteq S\} \\ &= \{S \subseteq \mathbb{P} \mid \forall Z \in \mathbb{P} . (Z \in \{E \in \mathbb{P} \mid \exists E' \in S . E \xrightarrow{a} E'\} \Rightarrow Z \in S)\} \\ &= \{S \subseteq \mathbb{P} \mid \forall E \in \mathbb{P} . ((\exists E' \in S . E \xrightarrow{a} E') \Rightarrow E \in S)\} \end{aligned}$$

which can be characterized by predicate

$$(PRE) \quad (\exists E' \in S . E \xrightarrow{a} E') \Rightarrow E \in S \quad (\text{for all } E \in \mathbb{P})$$

Back to the example ...

The set of *pre-fixed points* of

$$|\langle a \rangle|$$

is

$$\begin{aligned} Pre &= \{S \subseteq \mathbb{P} \mid |\langle a \rangle|(S) \subseteq S\} \\ &= \{S \subseteq \mathbb{P} \mid \forall E \in \mathbb{P} . ((\exists E' \in S . E \xrightarrow{a} E') \Rightarrow E \in S)\} \end{aligned}$$

- Clearly, $\{A \triangleq a.A\} \in Pre$
- but $\emptyset \in Pre$ as well

Therefore, its *least* solution is

$$\bigcap Pre = \emptyset$$

Conclusion: taking the **meaning** of $X = \langle a \rangle X$ as the *least* solution of the equation leads us to equate it to *false*

... but there is another possibility ...

$S \in \mathcal{P}\mathbb{P}$ is a *post-fixed point* of

$$|\langle a \rangle|$$

iff

$$S \subseteq |\langle a \rangle|(S)$$

leading to the following set of *post-fixed points*

$$\begin{aligned} \text{Post} &= \{S \subseteq \mathbb{P} \mid S \subseteq \{E \in \mathbb{P} \mid \exists E' \in S . E \xrightarrow{a} E'\}\} \\ &= \{S \subseteq \mathbb{P} \mid \forall Z \in \mathbb{P} . (Z \in S \Rightarrow Z \in \{E \in \mathbb{P} \mid \exists E' \in S . E \xrightarrow{a} E'\})\} \\ &= \{S \subseteq \mathbb{P} \mid \forall E \in \mathbb{P} . (E \in S \Rightarrow \exists E' \in S . E \xrightarrow{a} E')\} \end{aligned}$$

(POST) If $E \in S$ then $E \xrightarrow{a} E'$ for some $E' \in S$ (for all $E \in P$)

- i.e., if $E \in S$ it can perform a and this ability is maintained in its continuation

... but there is another possibility ...

- i.e., if $E \in S$ it can perform a and this ability is maintained in its continuation
- the *greatest* subset of \mathbb{P} verifying this condition is the set of processes with at least an infinite computation

Conclusion: taking the **meaning** of $X = \langle a \rangle X$ as the *greatest* solution of the equation characterizes the property **occurrence of a is possible**

The general case

- The meaning (i.e., **set of processes**) of a formula $X \triangleq \phi X$ where X occurs free in ϕ
- is a *solution* of equation

$$X = f(X) \quad \text{with} \quad f(S) = \llbracket \{S/X\}\phi \rrbracket$$

in \mathcal{PP} , where $\llbracket . \rrbracket$ is extended to formulae with variables by $\llbracket X \rrbracket = X$

The general case

The Knaster-Tarski theorem gives precise characterizations of the

- *smallest* solution: the intersection of all S such that

$$\text{(PRE)} \quad \text{If } E \in f(S) \text{ then } E \in S$$

to be denoted by

$$\mu X . \phi$$

- *greatest* solution: the union of all S such that

$$\text{(POST)} \quad \text{If } E \in S \text{ then } E \in f(S)$$

to be denoted by

$$\nu X . \phi$$

In the previous example:

$$\nu X . \langle a \rangle \text{true}$$

$$\mu X . \langle a \rangle \text{true}$$

The general case

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In the previous **example**:

$$\nu X . \langle a \rangle \text{true}$$

$$\mu X . \langle a \rangle \text{true}$$

The modal μ -calculus: syntax

... Hennessy-Milner + **recursion** (i.e. fixed points):

$$\phi ::= X \mid \phi_1 \wedge \phi_2 \mid \phi_1 \vee \phi_2 \mid \langle K \rangle \phi \mid [K] \phi \mid \mu X . \phi \mid \nu X . \phi$$

where $K \subseteq Act$ and X is a set of propositional variables

- Note that

$$true \stackrel{abv}{=} \nu X . X \quad \text{and} \quad false \stackrel{abv}{=} \mu X . X$$

The modal μ -calculus: denotational semantics

- Presence of variables requires models parametric on *valuations*:

$$V : X \rightarrow \mathcal{P}\mathcal{P}$$

- Then,

$$|X|_V = V(X)$$

$$|\phi_1 \wedge \phi_2|_V = |\phi_1|_V \cap |\phi_2|_V$$

$$|\phi_1 \vee \phi_2|_V = |\phi_1|_V \cup |\phi_2|_V$$

$$|[K]\phi|_V = |[K]|(|\phi|_V)$$

$$|\langle K \rangle \phi|_V = |\langle K \rangle|(|\phi|_V)$$

- and add

$$|\nu X . \phi|_V = \bigcup \{S \in \mathcal{P} \mid S \subseteq |[S/X]\phi|_V\}$$

$$|\mu X . \phi|_V = \bigcap \{S \in \mathcal{P} \mid |[S/X]\phi|_V \subseteq S\}$$

Notes

where

$$[[K]]X = \{F \in \mathbb{P} \mid \text{if } F \xrightarrow{a} F' \wedge a \in K \text{ then } F' \in X\}$$

$$|\langle K \rangle|X = \{F \in \mathbb{P} \mid \exists_{F' \in X, a \in K} . F \xrightarrow{a} F'\}$$

Modal μ -calculus

Intuition

- look at modal formulas as set-theoretic combinators
- introduce mechanisms to specify their fixed points
- introduced as a generalisation of Hennessy-Milner logic for processes to capture **enduring** properties.

References

- **Original reference:** *Results on the propositional μ -calculus*, D. Kozen, 1983.
- **Introductory text:** *Modal and temporal logics for processes*, C. Stirling, 1996

Notes

The modal μ -calculus [Kozen, 1983] is

- *decidable*
- strictly *more expressive* than PDL and CTL*

Moreover

- The *correspondence theorem* of the induced **temporal logic** with **bisimilarity** is kept

Example 1: $X \triangleq \phi \vee \langle a \rangle X$

Look for fixed points of

$$f(X) \triangleq |\phi| \cup |\langle a \rangle|(X)$$

Example 1: $X \triangleq \phi \vee \langle a \rangle X$

- (PRE) If $E \in f(X)$ then $E \in X$
- \equiv If $E \in (|\phi| \cup |\langle a \rangle|(X))$ then $E \in X$
- \equiv If $E \in \{F \mid F \models \phi\} \cup \{F \in \mathbb{P} \mid \exists F' \in X . F \xrightarrow{a} F'\}$
then $E \in X$
- \equiv if $E \models \phi \vee \exists E' \in X . E \xrightarrow{a} E'$ then $E \in X$

The *smallest* set of processes verifying this condition is composed of processes with at least a computation along which a can occur *until* ϕ holds. Taking its *intersection*, we end up with processes in which ϕ holds in a *finite* number of steps.

Example 1: $X \triangleq \phi \vee \langle a \rangle X$

$$\begin{aligned}
 (\text{POST}) \quad & \text{If } E \in X \text{ then } E \in f(X) \\
 \equiv \quad & \text{If } E \in X \text{ then } E \in (|\phi| \cup |\langle a \rangle|(X)) \\
 \equiv \quad & \text{If } E \in X \text{ then } E \in \{F \mid F \models \phi\} \cup \{F \in X \mid \exists F' \in X . F \xrightarrow{a} F'\} \\
 \equiv \quad & \text{If } E \in X \text{ then } E \models \phi \vee \exists E' \in X . E \xrightarrow{a} E'
 \end{aligned}$$

The *greatest* fixed point also includes processes which keep the possibility of doing a without ever reaching a state where ϕ holds.

Example 1: $X \triangleq \phi \vee \langle a \rangle X$

- *strong until*:

$$\mu X . \phi \vee \langle a \rangle X$$

- *weak until*

$$\nu X . \phi \vee \langle a \rangle X$$

Relevant particular cases:

- ϕ holds after internal activity:

$$\mu X . \phi \vee \langle \tau \rangle X$$

- ϕ holds in a finite number of steps

$$\mu X . \phi \vee \langle - \rangle X$$

Example 2: $X \triangleq \phi \wedge \langle a \rangle X$

(PRE) If $E \models \phi \wedge \exists_{E' \in X} . E \xrightarrow{a} E'$ then $E \in X$

implies that

$$\mu X . \phi \wedge \langle a \rangle X \Leftrightarrow \text{false}$$

(POST) If $E \in X$ then $E \models \phi \wedge \exists_{E' \in X} . E \xrightarrow{a} E'$

implies that

$$\nu X . \phi \wedge \langle a \rangle X$$

denote all processes which verify ϕ and have an *infinite* computation

Example 2: $X \triangleq \phi \wedge \langle a \rangle X$

Variant:

- ϕ holds along a finite or infinite a -computation:

$$\forall X . \phi \wedge (\langle a \rangle X \vee [a] \text{false})$$

In general:

- *weak safety*:

$$\forall X . \phi \wedge (\langle K \rangle X \vee [K] \text{false})$$

- *weak safety*, for $K = \text{Act}$:

$$\forall X . \phi \wedge (\langle - \rangle X \vee [-] \text{false})$$

Example 3: $X \triangleq [-]X$

(POST) If $E \in X$ then $E \in [[-]](X)$

\equiv If $E \in X$ then (if $E \xrightarrow{x} E'$ and $x \in Act$ then $E' \in X$)

implies $\nu X . [-]X \Leftrightarrow true$

(PRE) If (if $E \xrightarrow{x} E'$ and $x \in Act$ then $E' \in X$) then $E \in X$

implies $\mu X . [-]X$ represent *finite* processes (why?)

Safety and liveness

- *weak liveness*:

$$\mu X . \phi \vee \langle - \rangle X$$

- *strong safety*

$$\nu X . \psi \wedge [-] X$$

making $\psi = \neg\phi$ both properties are *dual*:

- there is at least a computation reaching a state s such that $s \models \phi$
- all states s reached along all computations maintain ϕ , ie, $s \models \neg\phi$

Safety and liveness

Qualifiers *weak* and *strong* refer to a **quantification over computations**

- *weak liveness*:

$$\mu X . \phi \vee \langle - \rangle X$$

(corresponds to Ctl formula $E F \phi$)

- *strong safety*

$$\nu X . \psi \wedge [-] X$$

(corresponds to Ctl formula $A G \psi$)

cf, liner time vs branching time

Duality

$$\neg(\mu X . \phi) = \nu X . \neg\phi$$

$$\neg(\nu X . \phi) = \mu X . \neg\phi$$

Example:

- *divergence*:

$$\nu X . \langle \tau \rangle X$$

- *convergence* (= all non observable behaviour is *finite*)

$$\neg(\nu X . \langle \tau \rangle X) = \mu X . \neg(\langle \tau \rangle X) = \mu X . [\tau]X$$

Safety and liveness

- *weak safety*:

$$\nu X . \phi \wedge (\langle - \rangle X \vee [-] \text{false})$$

(there is a computation along which ϕ holds)

- *strong liveness*

$$\mu X . \neg \phi \vee ([-] X \wedge \langle - \rangle \text{true})$$

(a state where the complement of ϕ holds can be *finitely* reached)

Conditional properties

$\phi_1 =$

After collecting a passenger (*icr*), the taxi drops him at destination (*fcr*)

Second part of ϕ_1 is **strong liveness**:

$$\mu X . [-fcr]X \wedge \langle - \rangle true$$

holding only after *icr*.

Is it enough to write:

$$[icr](\mu X . [-fcr]X \wedge \langle - \rangle true)$$

?

what we want does not depend on the initial state: it is *liveness embedded into strong safety*:

$$\forall Y . [icr](\mu X . [-fcr]X \wedge \langle - \rangle true) \wedge [-]Y$$

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$$\nu Y . [icr](\mu X . [-fcr]X \wedge \langle - \rangle true) \wedge [-]Y$$

Conditional properties

The previous example is *conditional liveness* but one can also have

- **conditional safety:**

$$\nu Y. (\neg\phi \vee (\phi \wedge \nu X. \psi \wedge [-]X)) \wedge [-]Y$$

(whenever ϕ holds, ψ cannot cease to hold)

Cyclic properties

$\phi =$ every second action is *out*

is expressed by

$$\forall X. [-]([-out]false \wedge [-]X)$$

$\phi =$ *out* follows *in*, but other actions can occur in between

$$\forall X. [out]false \wedge [in](\mu Y. [in]false \wedge [out]X \wedge [-out]Y) \wedge [-in]X$$

Note that the use of *least fixed points* imposes that the amount of computation between *in* and *out* is finite

Cyclic properties

$\phi =$ a state in which *in* can occur, can be reached an infinite number of times

$$\nu X . \mu Y . (\langle in \rangle true \vee \langle - \rangle Y) \wedge ([-] X \wedge \langle - \rangle true)$$

$\phi =$ *in* occurs an infinite number of times

$$\nu X . \mu Y . [-in] Y \wedge [-] X \wedge \langle - \rangle true$$

$\phi =$ *in* occurs a finite number of times

$$\mu X . \nu Y . [-in] Y \wedge [in] X$$