# Exercises 4 : Interaction and Concurrency

*Luís Soares Barbosa*

**Exercise** I.1

Suppose two variants of parallel composition have been added to the process language $\mathbb{P}$ and defined through the following rules:

$$\frac{E \xrightarrow{a} E'}{E \otimes F \xrightarrow{a} E' \otimes F}\ (O_1) \qquad \frac{F \xrightarrow{a} F'}{E \otimes F \xrightarrow{a} E \otimes F'}\ (O_2)$$

$$\frac{E \xrightarrow{a} E' \quad \wedge \quad \overline{a} \notin \mathcal{L}(F)}{E \parallel F \xrightarrow{a} E' \parallel F}\ (P_1) \qquad \frac{F \xrightarrow{a} F' \quad \wedge \quad \overline{a} \notin \mathcal{L}(E)}{E \parallel F \xrightarrow{a} E \parallel F'}\ (P_2)$$

$$\frac{E \xrightarrow{a} E' \quad F \xrightarrow{\overline{a}} F'}{E \parallel F \xrightarrow{\tau} E' \parallel F'}\ (P_3)$$

1. Explain, in your own words, the meaning of $\otimes$ e $\parallel$.
2. Guided by the semantic rules given, show how the synchronisation diagrams for $E \otimes F$ and $E \parallel F$ can be built from the corresponding diagrams for $E$ and $F$.
3. Is $\parallel$ associative with respect to $\sim$?

**Exercise** I.2

Identify, in the list of process pairs below, which of them can be related by $\approx$. And by $=$?

1. $a.\tau.b.\mathbf{0}$ e $a.b.\mathbf{0}$
2. $a.(b.\mathbf{0} + \tau.c.\mathbf{0})$ e $a.(b.\mathbf{0} + c.\mathbf{0})$
3. $a.(b.\mathbf{0} + \tau.c.\mathbf{0})$ e $a.(b.\mathbf{0} + c.\mathbf{0}) + a.c.\mathbf{0}$
4. $a.\mathbf{0} + b.\mathbf{0} + \tau.b.\mathbf{0}$ e $a.\mathbf{0} + \tau.b.\mathbf{0}$
5. $a.\mathbf{0} + b.\mathbf{0} + \tau.b.\mathbf{0}$ e $a.\mathbf{0} + b.\mathbf{0}$
6. $a.(b.\mathbf{0} + (\tau.(c.\mathbf{0} + \tau.d.\mathbf{0})))$ e $a.(b.\mathbf{0} + (\tau.(c.\mathbf{0} + \tau.d.\mathbf{0}))) + a.(c.\mathbf{0} + \tau.d.\mathbf{0})$
7. $a.(b.\mathbf{0} + (\tau.(c.\mathbf{0} + \tau.d.\mathbf{0})))$ e $a.(b.\mathbf{0} + c.\mathbf{0} + d.\mathbf{0}) + a.(c.\mathbf{0} + d.\mathbf{0}) + a.d.\mathbf{0}$
8. $\tau.(a.b.\mathbf{0} + a.c.\mathbf{0})$ e $\tau.a.b.\mathbf{0} + \tau.a.c.\mathbf{0}$
9. $\tau.(a.\tau.b.\mathbf{0} + a.b.\tau.\mathbf{0})$ e $a.b.\mathbf{0}$
10. $\tau.(\tau.a.\mathbf{0} + \tau.b.\mathbf{0})$ e $\tau.a.\mathbf{0} + \tau.b.\mathbf{0}$
11. $A \triangleq a.\tau.A$ e $B \triangleq a.B$
12. $A \triangleq \tau.A + a.\mathbf{0}$ e $a.\mathbf{0}$
13. $A \triangleq \tau.A$ e $\mathbf{0}$

**Exercise** I.3

Suppose processes $R$ and $T$ have transitions $R \xrightarrow{\tau} T$ and $T \xrightarrow{\tau} R$, among others. Show that, under this condition, $R = T$.

**Exercise** I.4

Consider the following statements about a binary relation $S$ on $\mathbb{P}$. Discuss whether you may conclude from each of them whether $S$ is (or is not) a weak bisimulation.

observacional:

1. $S$ is the identity in $\mathbb{P}$.
2. $S$ is a subset of the identity in $\mathbb{P}$.
3. $S$ is a strict bisimulation up to $\Leftrightarrow$.
4. $S$ is the empty relation.
5. $S = \{(a.E, a.F) \mid E \approx F\}$.
6. $S = \{(a.E, a.F) \mid E \approx F\} \cup \approx$.

**Exercise** I.5

Show that

1. $E + \tau.(E + F) \;=\; \tau.(E + F)$
2. $a.(E + \tau.\tau.E) \;=\; a.E$
3. $\tau.(G + a.(E + \tau.F)) \;=\; \tau.(G + a.(E + \tau.F)) + a.F$

**Exercise** I.6

Show that any process $\tau.(\tau.P + a.\mathbf{0})$ is a solution to equation $X \;=\; a.\mathbf{0} + \tau.X$.

**Exercise** I.7

Let $E$ be a process such that $\mathsf{fn}(E) = \emptyset$. Prove or refute the following statements:

1. $E \mid Q \approx Q$.
2. $E \mid Q = Q$.
3. $E \mid Q = \tau.Q$.

**Exercise** I.8

Although concurrent systems usually deal with components exhibiting non terminating behaviour, it is sometimes useful also to consider terminating processes and their composition. Let $T$ be a class of terminating processes which perform a special action † to announce completion of all their tasks and evolve to **0** after that. In this class it is possible to define a combinator for *sequential* composition $P\,;Q$, whose behaviour is informally explained as *once $P$ terminates, $P\,;Q$ behaves like $Q$*. Formally,

$$P\,;Q \;\triangleq\; \mathsf{new}\,\{m\}\;(\{m/\dagger\}\,P \mid \overline{m}\cdot Q)$$

where $m$ is fresh identifier, not occurring neither in $P$ nor $Q$.

1. Define a process $U \in T$ such that $U \,;\, P \approx P$. Justify your proposal.
2. Prove or refute that, for any $P, Q, R \in T$,
$$(P + Q) \,;\, R \;\approx\; (P \,;\, R) + (Q \,;\, R)$$
3. As sequential composition is a particular case of parallel composition, the law above could be regarded as a particular case of
$$(P + Q) \mid R \;\approx\; (P \mid R) + (Q \mid R)$$
This equation, however, is false. Confirm this by providing a suitable counter-example..

---

**Exercise** I.9

---

Consider the following specification of a *pipe*, as supported e.g. in UNIX:

$$U \rhd V \;\stackrel{\text{abv}}{=}\; \mathsf{new}\,\{c\}\ (\{c/out\}U \mid \{c/in\}V)$$

under the assumption that, in both processes, actions $\overline{out}$ e $in$ stand for, respectively, the output and input ports.

1. Consider now the following processes only partially defined:
$$U_1 \triangleq \overline{out}.T$$
$$V_1 \triangleq in.R$$
$$U_2 \triangleq \overline{out}.\overline{out}.\overline{out}.T$$
$$V_2 \triangleq in.in.in.R$$
Prove, by equational reasoning, or refute the following properties:
   (a) $U_1 \rhd V_1 \;\sim\; T \rhd R$
   (b) $U_2 \rhd V_2 \;=\; U_1 \rhd V_1$
2. Show or refute the associativity of $\rhd$ wrt process equality, *i.e.*, for all $P, T, V \in \mathbb{P}$,
$$(U \rhd V) \rhd T \;=\; U \rhd (V \rhd T)$$
3. Show that $\mathbf{0} \rhd \mathbf{0} \;=\; \mathbf{0}$.

---

**Exercise** I.10

---

Consider a combinator $\circlearrowleft_n$ whose operational semantics is given by following rule

$$\frac{E \xrightarrow{a} E'}{\circlearrowleft_0 E \xrightarrow{a} E'} \qquad \frac{E \xrightarrow{a} E'}{\circlearrowleft_n E \xrightarrow{a} \circlearrowleft_{n-1} E'} \quad \text{for } n > 0$$

1. Explain its purpose.
2. Discuss whether, and for which values of $m$ and $n$, one may have $\circlearrowleft_n (\circlearrowleft_m E) \sim \circlearrowleft_n E$.
3. Show that $E \sim F$ implies $\circlearrowleft_n E \sim \circlearrowleft_n F$.
4. Show, by a counter-example, that, whenever $\sim$ is replaced by $\approx$, the implication above fails.
5. How could the operational semantics of this new combinator be changed so that the implication mentioned above holds? I.e. so that $E \approx F \;\Rightarrow\; \circlearrowleft_n E \approx \circlearrowleft_n F$ ?

---

**Exercise** I.11

---

Consider a combinator whose operational semantics is given by following rule

$$\frac{E \xrightarrow{x} E'}{E \downarrow a \xrightarrow{x} E'} \text{ if } \; x \neq a, x \neq \overline{a}$$

1. Explain its purpose.
2. Show that $P \downarrow a \sim Q \downarrow a$ if $P \sim Q$.
3. Define two processes $E$ and $F$ such that $E \approx F$ but $E \downarrow a \not\approx F \downarrow a$.
4. Prove or refute that if $P = Q$ then $P \downarrow a = Q \downarrow a$.

---

**Exercise** I.12

---

Consider a new process combinator, called an *action duplicator*, and defined by the following rule:

$$\frac{E \xrightarrow{a} E'}{\circlearrowleft(E) \xrightarrow{a} E}$$

Note that the derivative in the rule's conclusion is $E$ (and not $E'$). For example, $\circlearrowleft(a.\mathbf{0}) \xrightarrow{a} a.\mathbf{0}$. Prove or refute that

1. $E \sim F$ implies $\circlearrowleft(E) \sim \circlearrowleft(F)$.
2. $E \approx F$ implies $\circlearrowleft(E) \approx \circlearrowleft(F)$.
3. $\circlearrowleft(E + F) \sim \circlearrowleft(E) + \circlearrowleft(F)$.