

# Chapter 2

## Physics, Topology, Logic and Computation: A Rosetta Stone

J. Baez and M. Stay

**Abstract** In physics, Feynman diagrams are used to reason about quantum processes. In the 1980s, it became clear that underlying these diagrams is a powerful analogy between quantum physics and topology. Namely, a linear operator behaves very much like a “cobordism”: a manifold representing spacetime, going between two manifolds representing space. This led to a burst of work on topological quantum field theory and “quantum topology”. But this was just the beginning: similar diagrams can be used to reason about logic, where they represent proofs, and computation, where they represent programs. With the rise of interest in quantum cryptography and quantum computation, it became clear that there is extensive network of analogies between physics, topology, logic and computation. In this expository paper, we make some of these analogies precise using the concept of “closed symmetric monoidal category”. We assume no prior knowledge of category theory, proof theory or computer science.

### 2.1 Introduction

Category theory is a very general formalism, but there is a certain special way that physicists use categories which turns out to have close analogues in topology, logic and computation. A category has *objects* and *morphisms*, which represent *things* and *ways to go between things*. In physics, the objects are often *physical systems*, and the morphisms are *processes* turning a state of one physical system into a state of another system—perhaps the same one. In quantum physics we often formalize this by taking *Hilbert spaces* as objects, and *linear operators* as morphisms.

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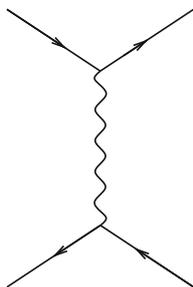
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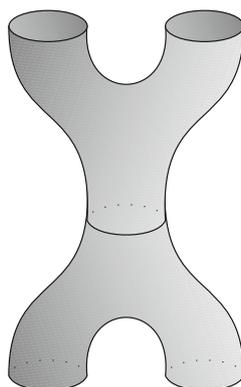
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Sometime around 1949, Feynman [63] realized that in quantum field theory it is useful to draw linear operators as diagrams:



This lets us reason with them pictorially. We can warp a picture without changing the operator it stands for: all that matters is the topology, not the geometry. In the 1970s, Penrose realized that generalizations of Feynman diagrams arise throughout quantum theory, and might even lead to revisions in our understanding of spacetime [84–87]. In the 1980s, it became clear that underlying these diagrams is a powerful analogy between quantum physics and topology! Namely, a linear operator behaves very much like a “cobordism”—that is, an  $n$ -dimensional manifold going between manifolds of one dimension less:



String theory exploits this analogy by replacing the Feynman diagrams of ordinary quantum field theory with 2-dimensional cobordisms, which represent the world-sheets traced out by strings with the passage of time. The analogy between operators and cobordisms is also important in loop quantum gravity and—most of all—the more purely mathematical discipline of “topological quantum field theory”.

Meanwhile, quite separately, logicians had begun using categories where the objects represent *propositions* and the morphisms represent *proofs*. The idea is that a proof is a process going from one proposition (the hypothesis) to another (the conclusion). Later, computer scientists started using categories where the objects represent *data types* and the morphisms represent *programs*. They also started using “flow charts” to describe programs. Abstractly, these are very much like Feynman diagrams!

The logicians and computer scientists were never very far from each other. Indeed, the “Curry–Howard correspondence” relating proofs to programs has been well-known at least since the early 1970s, with roots stretching back earlier

[36, 37, 56]. But, it is only in the 1990s that the logicians and computer scientists bumped into the physicists and topologists. One reason is the rise of interest in quantum cryptography and quantum computation [29]. With this, people began to think of quantum processes as forms of information processing, and apply ideas from computer science. It was then realized that the loose analogy between flow charts and Feynman diagrams could be made more precise and powerful with the aid of category theory [3].

By now there is an extensive network of interlocking analogies between physics, topology, logic and computer science. They suggest that research in the area of common overlap is actually trying to build a new science: *a general science of systems and processes*. Building this science will be very difficult. There are good reasons for this, but also bad ones. One bad reason is that different fields use different terminology and notation.

The original Rosetta Stone, created in 196 BC, contains versions of the same text in three languages: demotic Egyptian, hieroglyphic script and classical Greek. Its rediscovery by Napoleon’s soldiers let modern Egyptologists decipher the hieroglyphs. Eventually this led to a vast increase in our understanding of Egyptian culture.

At present, the deductive systems in mathematical logic look like hieroglyphs to most physicists. Similarly, quantum field theory is Greek to most computer scientists, and so on. So, there is a need for a new Rosetta Stone to aid researchers attempting to translate between fields. Table 2.1 shows our guess as to what this Rosetta Stone might look like.

**Table 2.1** The Rosetta Stone (pocket version)

Category theory	Physics	Topology	Logic	Computation
Object	System	Manifold	Proposition	Data type
Morphism	Process	Cobordism	Proof	Program

The rest of this paper expands on this table by comparing how categories are used in physics, topology, logic, and computation. Unfortunately, these different fields focus on slightly different kinds of categories. Though most physicists don’t know it, quantum physics has long made use of “compact symmetric monoidal categories”. Knot theory uses “compact braided monoidal categories”, which are slightly more general. However, it became clear in the 1990s that these more general gadgets are useful in physics too. Logic and computer science used to focus on “cartesian closed categories”—where “cartesian” can be seen, roughly, as an antonym of “quantum”. However, thanks to work on linear logic and quantum computation, some logicians and computer scientists have dropped their insistence on cartesianness: now they study more general sorts of “closed symmetric monoidal categories”.

In Sect. 2.2 we explain these concepts, how they illuminate the analogy between physics and topology, and how to work with them using string diagrams. We assume no prior knowledge of category theory, only a willingness to learn some. In Sect. 2.3 we explain how closed symmetric monoidal categories correspond to a small fragment of ordinary propositional logic, which also happens to be a fragment of Girard’s “linear logic” [47]. In Sect. 2.4 we explain how closed symmetric monoidal

categories correspond to a simple model of computation. Each of these sections starts with some background material. In Sect. 2.5, we conclude by presenting a larger version of the Rosetta Stone.

Our treatment of all four subjects—physics, topology, logic and computation—is bound to seem sketchy, narrowly focused and idiosyncratic to practitioners of these subjects. Our excuse is that we wish to emphasize certain analogies while saying no more than absolutely necessary. To make up for this, we include many references for those who wish to dig deeper.

## 2.2 The Analogy Between Physics and Topology

### 2.2.1 Background

Currently our best theories of physics are general relativity and the Standard Model of particle physics. The first describes gravity without taking quantum theory into account; the second describes all the other forces taking quantum theory into account, but ignores gravity. So, our world-view is deeply schizophrenic. The field where physicists struggle to solve this problem is called *quantum gravity*, since it is widely believed that the solution requires treating gravity in a way that takes quantum theory into account.

**Table 2.2** Analogy between physics and topology

Physics	Topology
Hilbert space (system)	$(n - 1)$ -Dimensional manifold (space)
Operator between Hilbert spaces (process)	Cobordism between $(n - 1)$ -dimensional manifolds (spacetime)
Composition of operators	Composition of cobordisms
Identity operator	Identity cobordism

Nobody is sure how to do this, but there is a striking similarity between two of the main approaches: string theory and loop quantum gravity. Both rely on the analogy between physics and topology shown in Table 2.2. On the left we have a basic ingredient of quantum theory: the category  $\text{Hilb}$  whose objects are Hilbert spaces, used to describe physical *systems*, and whose morphisms are linear operators, used to describe physical *processes*. On the right we have a basic structure in differential topology: the category  $n\text{Cob}$ . Here the objects are  $(n-1)$ -dimensional manifolds, used to describe *space*, and whose morphisms are  $n$ -dimensional cobordisms, used to describe *spacetime*.

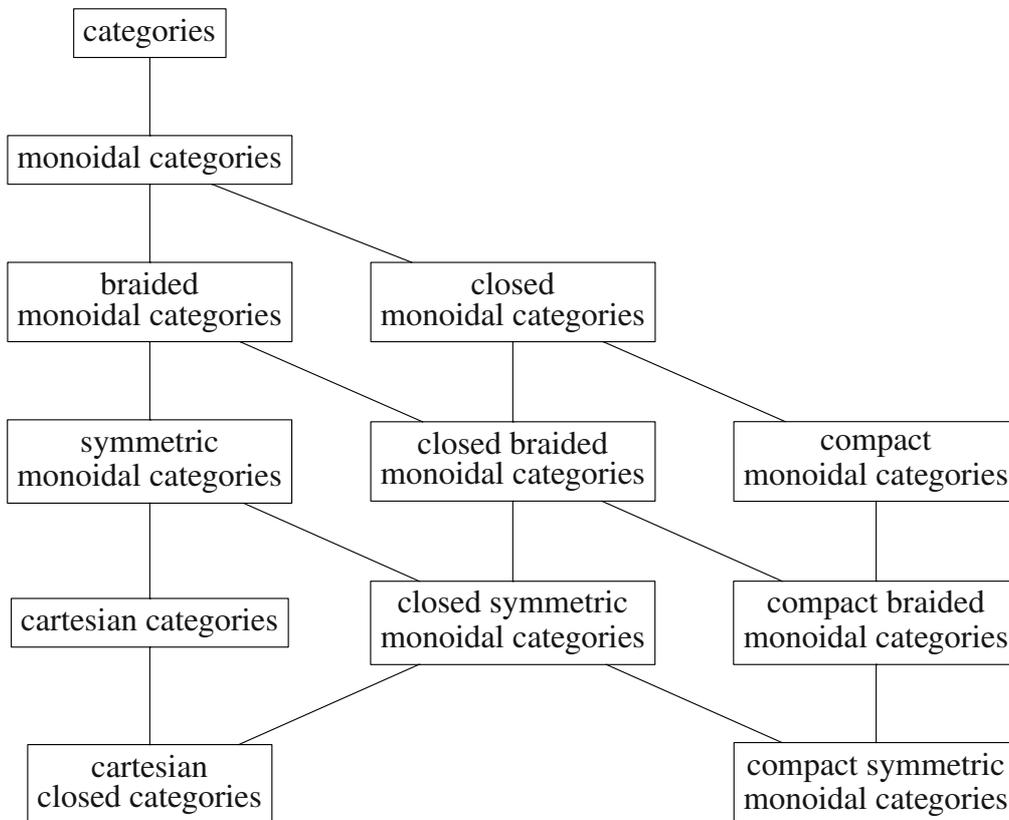
As we shall see,  $\text{Hilb}$  and  $n\text{Cob}$  share many structural features. Moreover, both are very different from the more familiar category  $\text{Set}$ , whose objects are sets and whose morphisms are functions. Elsewhere we have argued at great length that this

is important for better understanding quantum gravity [11] and even the foundations of quantum theory [12]. The idea is that if Hilb is more like  $n\text{Cob}$  than Set, maybe we should stop thinking of a quantum process as a function from one set of states to another. Instead, maybe we should think of it as resembling a “spacetime” going between spaces of dimension one less.

This idea sounds strange, but the simplest example is something very practical, used by physicists every day: a Feynman diagram. This is a 1-dimensional graph going between 0-dimensional collections of points, with edges and vertices labelled in certain ways. Feynman diagrams are topological entities, but they describe linear operators. String theory uses 2-dimensional cobordisms equipped with extra structure—string worldsheets—to do a similar job. Loop quantum gravity uses 2d generalizations of Feynman diagrams called “spin foams” [10]. Topological quantum field theory uses higher-dimensional cobordisms [14]. In each case, processes are described by morphisms in a special sort of category: a “compact symmetric monoidal category”.

In what follows, we shall not dwell on puzzles from quantum theory or quantum gravity. Instead we take a different tack, simply explaining some basic concepts from category theory and showing how Set, Hilb,  $n\text{Cob}$  and categories of tangles give examples. A recurring theme, however, is that Set is very different from the other examples.

To help the reader safely navigate the sea of jargon, here is a chart of the concepts we shall explain in this section:



The category Set is cartesian closed, while Hilb and  $n\text{Cob}$  are compact symmetric monoidal.

## 2.2.2 Categories

Category theory was born around 1945, with Eilenberg and Mac Lane [40] defining “categories”, “functors” between categories, and “natural transformations” between functors. By now there are many introductions to the subject [35, 78, 81], including some available for free online [21, 50]. Nonetheless, we begin at the beginning:

**Definition 1** A **category**  $C$  consists of:

- a collection of **objects**, where if  $X$  is an object of  $C$  we write  $X \in C$ , and
- for every pair of objects  $(X, Y)$ , a set  $\text{hom}(X, Y)$  of **morphisms** from  $X$  to  $Y$ . We call this set  $\text{hom}(X, Y)$  a **homset**. If  $f \in \text{hom}(X, Y)$ , then we write  $f : X \rightarrow Y$ .

such that:

- for every object  $X$  there is an **identity morphism**  $1_X : X \rightarrow X$ ;
- morphisms are composable: given  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ , there is a **composite morphism**  $gf : X \rightarrow Z$ ; sometimes also written  $g \circ f$ .
- an identity morphism is both a **left and a right unit** for composition: if  $f : X \rightarrow Y$ , then  $f1_X = f = 1_Y f$ ; and
- composition is **associative**:  $(hg)f = h(gf)$  whenever either side is well-defined.

**Definition 2** We say a morphism  $f : X \rightarrow Y$  is an **isomorphism** if it has an inverse—that is, there exists another morphism  $g : Y \rightarrow X$  such that  $gf = 1_X$  and  $fg = 1_Y$ .

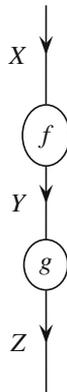
A category is the simplest framework where we can talk about systems (objects) and processes (morphisms). To visualize these, we can use “Feynman diagrams” of a very primitive sort. In applications to linear algebra, these diagrams are often called “spin networks”, but category theorists call them “string diagrams”, and that is the term we will use. The term “string” here has little to do with string theory: instead, the idea is that objects of our category label “strings” or “wires”:



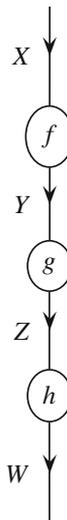
and morphisms  $f : X \rightarrow Y$  label “black boxes” with an input wire of type  $X$  and an output wire of type  $Y$ :



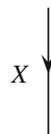
We compose two morphisms by connecting the output of one black box to the input of the next. So, the composite of  $f: X \rightarrow Y$  and  $g: Y \rightarrow Z$  looks like this:



Associativity of composition is then implicit:



is our notation for both  $h(gf)$  and  $(hg)f$ . Similarly, if we draw the identity morphism  $1_X: X \rightarrow X$  as a piece of wire of type  $X$ :



then the left and right unit laws are also implicit.

There are countless examples of categories, but we will focus on four:

- **Set**: the category where objects are sets.
- **Hilb**: the category where objects are finite-dimensional Hilbert spaces.
- **$n\text{Cob}$** : the category where morphisms are  $n$ -dimensional cobordisms.
- **$\text{Tang}_k$** : the category where morphisms are  $k$ -codimensional tangles.

As we shall see, all four are closed symmetric monoidal categories, at least when  $k$  is big enough. However, the most familiar of the lot, namely  $\text{Set}$ , is the odd man out: it is “cartesian”.

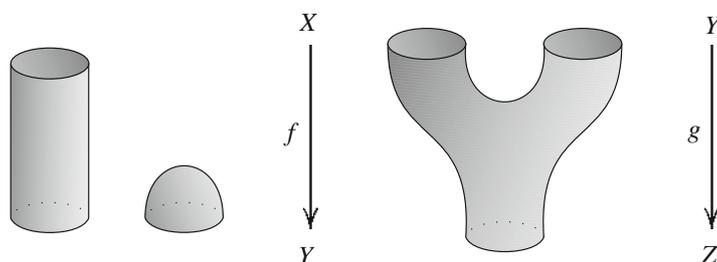
Traditionally, mathematics has been founded on the category  $\text{Set}$ , where the objects are *sets* and the morphisms are *functions*. So, when we study systems and processes in physics, it is tempting to specify a system by giving its set of states, and a process by giving a function from states of one system to states of another.

However, in quantum physics we do something subtly different: we use categories where objects are *Hilbert spaces* and morphisms are *bounded linear operators*. We specify a system by giving a Hilbert space, but this Hilbert space is not really the set of states of the system: a state is actually a ray in Hilbert space. Similarly, a bounded linear operator is not precisely a function from states of one system to states of another.

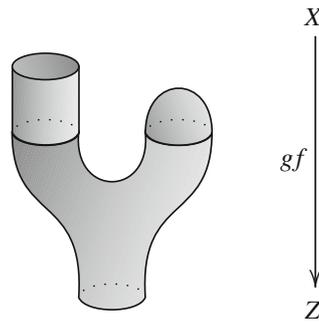
In the day-to-day practice of quantum physics, what really matters is not sets of states and functions between them, but Hilbert space and operators. One of the virtues of category theory is that it frees us from the “Set-centric” view of traditional mathematics and lets us view quantum physics on its own terms. As we shall see, this sheds new light on the quandaries that have always plagued our understanding of the quantum realm [12].

To avoid technical issues that would take us far afield, we will take  $\text{Hilb}$  to be the category where objects are *finite-dimensional Hilbert spaces* and morphisms are *linear operators* (automatically bounded in this case). Finite-dimensional Hilbert spaces suffice for some purposes; infinite-dimensional ones are often important, but treating them correctly would require some significant extensions of the ideas we want to explain here.

In physics we also use categories where the objects represent choices of *space*, and the morphisms represent choices of *spacetime*. The simplest is  $n\text{Cob}$ , where the objects are  $(n - 1)$ -dimensional manifolds, and the morphisms are  $n$ -dimensional cobordisms. Glossing over some subtleties that a careful treatment would discuss [90], a cobordism  $f: X \rightarrow Y$  is an  $n$ -dimensional manifold whose boundary is the disjoint union of the  $(n - 1)$ -dimensional manifolds  $X$  and  $Y$ . Here are a couple of cobordisms in the case  $n = 2$ :



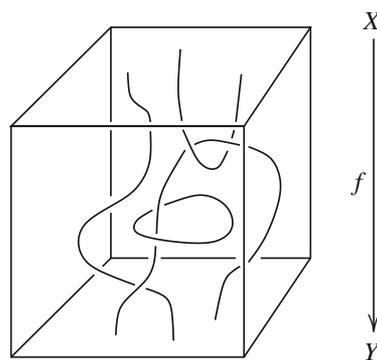
We compose them by gluing the “output” of one to the “input” of the other. So, in the above example  $gf: X \rightarrow Z$  looks like this:



Another kind of category important in physics has objects representing *collections of particles*, and morphisms representing their *worldlines and interactions*. Feynman diagrams are the classic example, but in these diagrams the “edges” are not taken literally as particle trajectories. An example with closer ties to topology is  $\text{Tang}_k$ .

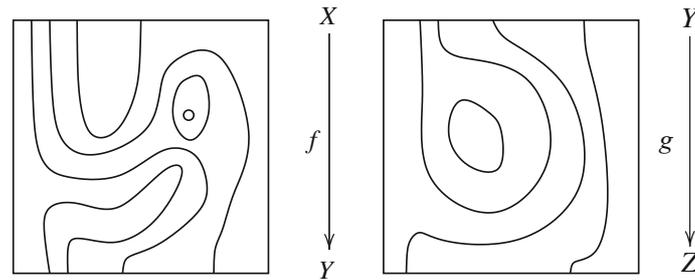
Very roughly speaking, an object in  $\text{Tang}_k$  is a collection of points in a  $k$ -dimensional cube, while a morphism is a “tangle”: a collection of arcs and circles smoothly embedded in a  $(k + 1)$ -dimensional cube, such that the circles lie in the interior of the cube, while the arcs touch the boundary of the cube only at its top and bottom, and only at their endpoints. A bit more precisely, tangles are “isotopy classes” of such embedded arcs and circles: this equivalence relation means that only the topology of the tangle matters, not its geometry. We compose tangles by attaching one cube to another top to bottom.

More precise definitions can be found in many sources, at least for  $k = 2$ , which gives tangles in a 3-dimensional cube [46, 64, 90, 99, 107, 111]. But since a picture is worth a thousand words, here is a picture of a morphism in  $\text{Tang}_2$ :

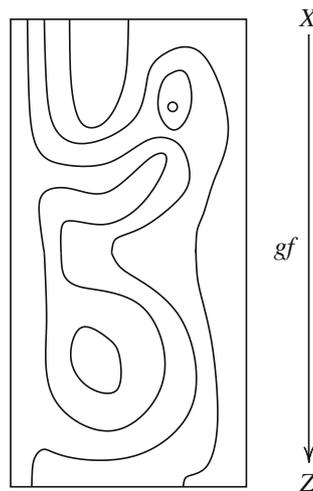


Note that we can think of a morphism in  $\text{Tang}_k$  as a 1-dimensional cobordism *embedded in a  $k$ -dimensional cube*. This is why  $\text{Tang}_k$  and  $n\text{Cob}$  behave similarly in some respects.

Here are two composable morphisms in  $\text{Tang}_1$ :



and here is their composite:



Since only the tangle's topology matters, we are free to squash this rectangle into a square if we want, but we do not need to.

It is often useful to consider tangles that are decorated in various ways. For example, in an “oriented” tangle, each arc and circle is equipped with an orientation. We can indicate this by drawing a little arrow on each curve in the tangle. In applications to physics, these curves represent worldlines of particles, and the arrows say whether each particle is going forwards or backwards in time, following Feynman’s idea that antiparticles are particles going backwards in time. We can also consider “framed” tangles. Here each curve is replaced by a “ribbon”. In applications to physics, this keeps track of how each particle twists. This is especially important for fermions, where a  $2\pi$  twist acts nontrivially. Mathematically, the best-behaved tangles are both framed and oriented [14, 99], and these are what we should use to define  $\text{Tang}_k$ . The category  $n\text{Cob}$  also has a framed oriented version. However, these details will barely matter in what is to come.

It is difficult to do much with categories without discussing the maps between them. A map between categories is called a ‘functor’:

**Definition 3** A **functor**  $F: C \rightarrow D$  from a category  $C$  to a category  $D$  is a map sending:

- any object  $X \in C$  to an object  $F(X) \in D$ ,
- any morphism  $f: X \rightarrow Y$  in  $C$  to a morphism  $F(f): F(X) \rightarrow F(Y)$  in  $D$ ,

in such a way that:

- **$F$  preserves identities:** for any object  $X \in C$ ,  $F(1_X) = 1_{F(X)}$ ;
- **$F$  preserves composition:** for any pair of morphisms  $f: X \rightarrow Y$ ,  $g: Y \rightarrow Z$  in  $C$ ,  $F(gf) = F(g)F(f)$ .

In the sections to come, we will see that functors and natural transformations are useful for putting extra structure on categories. Here is a rather different use for functors: we can think of a functor  $F: C \rightarrow D$  as giving a picture, or “representation”, of  $C$  in  $D$ . The idea is that  $F$  can map objects and morphisms of some ‘abstract’ category  $C$  to objects and morphisms of a more “concrete” category  $D$ .

For example, consider an abstract group  $G$ . This is the same as a category with one object and with all morphisms invertible. The object is uninteresting, so we can just call it  $\bullet$ , but the morphisms are the elements of  $G$ , and we compose them by multiplying them. From this perspective, a **representation** of  $G$  on a finite-dimensional Hilbert space is the same as a functor  $F: G \rightarrow \text{Hilb}$ . Similarly, an **action** of  $G$  on a set is the same as a functor  $F: G \rightarrow \text{Set}$ . Both notions are ways of making an abstract group more concrete.

Ever since Lawvere’s 1963 thesis on functorial semantics [75], the idea of functors as representations has become pervasive. However, the terminology varies from field to field. Following Lawvere, logicians often call the category  $C$  a “theory”, and call the functor  $F: C \rightarrow D$  a “model” of this theory. Other mathematicians might call  $F$  an “algebra” of the theory. In this work, the default choice of  $D$  is usually the category  $\text{Set}$ .

In physics, it is the functor  $F: C \rightarrow D$  that is called the “theory”. Here the default choice of  $D$  is either the category we are calling  $\text{Hilb}$  or a similar category of *infinite-dimensional* Hilbert spaces. For example, both “conformal field theories” [95] and topological quantum field theories [8, 9] can be seen as functors of this sort.

If we think of functors as models, natural transformations are maps between models:

**Definition 4** Given two functors  $F, F': C \rightarrow D$ , a **natural transformation**  $\alpha: F \Rightarrow F'$  assigns to every object  $X$  in  $C$  a morphism  $\alpha_X: F(X) \rightarrow F'(X)$  such that for any morphism  $f: X \rightarrow Y$  in  $C$ , the equation  $\alpha_Y F(f) = F'(f) \alpha_X$  holds in  $D$ . In other words, this square commutes:

$$\begin{array}{ccc}
 F(X) & \xrightarrow{F(f)} & F(Y) \\
 \alpha_X \downarrow & & \downarrow \alpha_Y \\
 F'(X) & \xrightarrow{F'(f)} & F'(Y)
 \end{array}$$

(Going across and then down equals going down and then across.)

**Definition 5** A **natural isomorphism** between functors  $F, F': C \rightarrow D$  is a natural transformation  $\alpha: F \Rightarrow F'$  such that  $\alpha_X$  is an isomorphism for every  $X \in C$ .

For example, suppose  $F, F': G \rightarrow \text{Hilb}$  are functors where  $G$  is a group, thought of as a category with one object, say  $\bullet$ . Then, as already mentioned,  $F$  and  $F'$  are secretly just representations of  $G$  on the Hilbert spaces  $F(\bullet)$  and  $F'(\bullet)$ . A natural transformation  $\alpha: F \Rightarrow F'$  is then the same as an **intertwining operator** from one representation to another: that is, a linear operator

$$A: F(\bullet) \rightarrow F'(\bullet)$$

satisfying

$$AF(g) = F'(g)A$$

for all group elements  $g$ .

### 2.2.3 Monoidal Categories

In physics, it is often useful to think of two systems sitting side by side as forming a single system. In topology, the disjoint union of two manifolds is again a manifold in its own right. In logic, the conjunction of two statements is again a statement. In programming we can combine two data types into a single “product type”. The concept of “monoidal category” unifies all these examples in a single framework.

A monoidal category  $C$  has a functor  $\otimes: C \times C \rightarrow C$  that takes two objects  $X$  and  $Y$  and puts them together to give a new object  $X \otimes Y$ . To make this precise, we need the cartesian product of categories:

**Definition 6** The **cartesian product**  $C \times C'$  of categories  $C$  and  $C'$  is the category where:

- an object is a pair  $(X, X')$  consisting of an object  $X \in C$  and an object  $X' \in C'$ ;
- a morphism from  $(X, X')$  to  $(Y, Y')$  is a pair  $(f, f')$  consisting of a morphism  $f: X \rightarrow Y$  and a morphism  $f': X' \rightarrow Y'$ ;
- composition is done componentwise:  $(g, g')(f, f') = (gf, g'f')$ ;
- identity morphisms are defined componentwise:  $1_{(X, X')} = (1_X, 1_{X'})$ .

Mac Lane [77] defined monoidal categories in 1963. The subtlety of the definition lies in the fact that  $(X \otimes Y) \otimes Z$  and  $X \otimes (Y \otimes Z)$  are not usually equal. Instead, we should specify an isomorphism between them, called the “associator”. Similarly, while a monoidal category has a “unit object”  $I$ , it is not usually true that  $I \otimes X$  and  $X \otimes I$  equal  $X$ . Instead, we should specify isomorphisms  $I \otimes X \cong X$  and  $X \otimes I \cong X$ . To be manageable, all these isomorphisms must then satisfy certain equations:

**Definition 7** A **monoidal category** consists of:

- a category  $C$ ,
- a **tensor product** functor  $\otimes: C \times C \rightarrow C$ ,

- a **unit object**  $I \in C$ ,
- a natural isomorphism called the **associator**, assigning to each triple of objects  $X, Y, Z \in C$  an isomorphism

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z),$$

- natural isomorphisms called the **left** and **right unitors**, assigning to each object  $X \in C$  isomorphisms

$$l_X : I \otimes X \xrightarrow{\sim} X$$

$$r_X : X \otimes I \xrightarrow{\sim} X,$$

such that:

- for all  $X, Y \in C$  the **triangle equation** holds:

$$\begin{array}{ccc}
 (X \otimes I) \otimes Y & \xrightarrow{a_{X,I,Y}} & X \otimes (I \otimes Y) \\
 \searrow r_X \otimes 1_Y & & \swarrow 1_X \otimes 1_Y \\
 & X \otimes Y &
 \end{array}$$

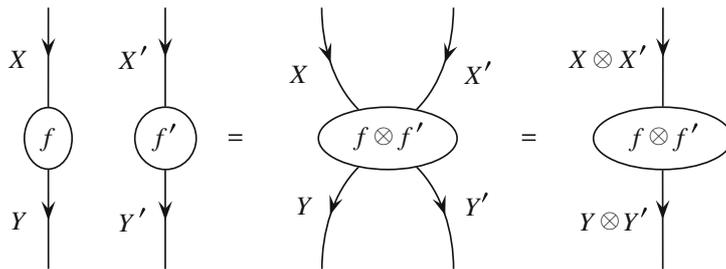
- for all  $W, X, Y, Z \in C$ , the **pentagon equation** holds:

$$\begin{array}{ccccc}
 & & ((W \otimes X) \otimes Y) \otimes Z & & \\
 & & \swarrow & \searrow^{a_{W,X,Y} \otimes 1_Z} & \\
 & & & & (W \otimes (X \otimes Y)) \otimes Z \\
 & & & & \downarrow a_{W,X \otimes Y,Z} \\
 & & & & W \otimes ((X \otimes Y) \otimes Z) \\
 & & & \swarrow^{1_W \otimes a_{X,Y,Z}} & \\
 & & & & W \otimes (X \otimes (Y \otimes Z)) \\
 & & \swarrow^{a_{W \otimes X,Y,Z}} & \searrow^{a_{W,X,Y \otimes Z}} & \\
 & & (W \otimes X) \otimes (Y \otimes Z) & &
 \end{array}$$

When we have a tensor product of four objects, there are five ways to parenthesize it, and at first glance the associator lets us build two isomorphisms from  $W \otimes (X \otimes (Y \otimes Z))$  to  $((W \otimes X) \otimes Y) \otimes Z$ . But, the pentagon equation says these

isomorphisms are equal. When we have tensor products of even more objects there are even more ways to parenthesize them, and even more isomorphisms between them built from the associator. However, Mac Lane showed that the pentagon identity implies these isomorphisms are all the same. Similarly, if we also assume the triangle equation, all isomorphisms with the same source and target built from the associator, left and right unit laws are equal.

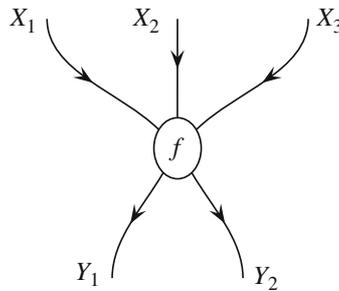
In a monoidal category we can do processes in “parallel” as well as in “series”. Doing processes in series is just composition of morphisms, which works in any category. But in a monoidal category we can also tensor morphisms  $f : X \rightarrow Y$  and  $f' : X' \rightarrow Y'$  and obtain a “parallel process”  $f \otimes f' : X \otimes X' \rightarrow Y \otimes Y'$ . We can draw this in various ways:



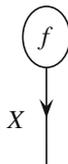
More generally, we can draw any morphism

$$f : X_1 \otimes \cdots \otimes X_n \rightarrow Y_1 \otimes \cdots \otimes Y_m$$

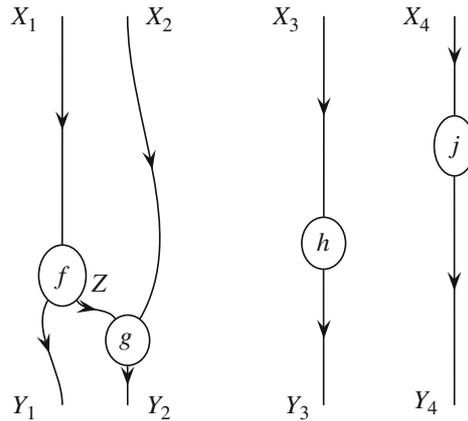
as a black box with  $n$  input wires and  $m$  output wires:



We draw the unit object  $I$  as a blank space. So, for example, we draw a morphism  $f : I \rightarrow X$  as follows:

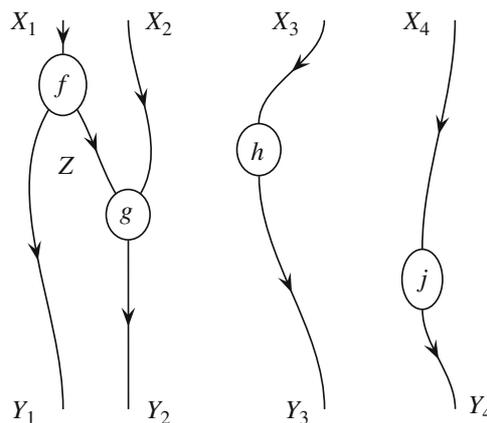


By composing and tensoring morphisms, we can build up elaborate pictures resembling Feynman diagrams:

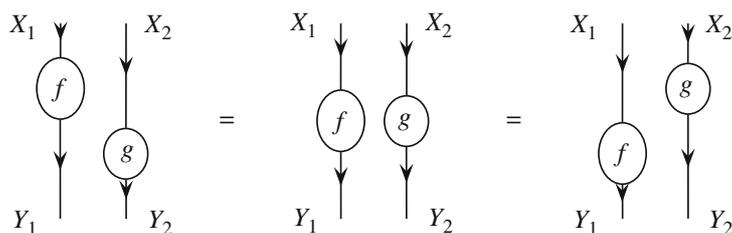


The laws governing a monoidal category allow us to neglect associators and unitors when drawing such pictures, without getting in trouble. The reason is that Mac Lane’s Coherence Theorem says any monoidal category is “equivalent”, in a suitable sense, to one where all associators and unitors are identity morphisms [77].

We can also deform the picture in a wide variety of ways without changing the morphism it describes. For example, the above morphism equals this one:



Everyone who uses string diagrams for calculations in monoidal categories starts by worrying about the rules of the game: *precisely how* can we deform these pictures without changing the morphisms they describe? Instead of stating the rules precisely—which gets a bit technical—we urge you to explore for yourself what is allowed and what is not. For example, show that we can slide black boxes up and down like this:



For a formal treatment of the rules governing string diagrams, try the original papers by Joyal and Street [59, 60] and the book by Yetter [111].

Now let us turn to examples. Here it is crucial to realize that the same category can often be equipped with different tensor products, resulting in different monoidal categories:

- There is a way to make  $\text{Set}$  into a monoidal category where  $X \otimes Y$  is the cartesian product  $X \times Y$  and the unit object is any one-element set. Note that this tensor product is not strictly associative, since  $(x, (y, z)) \neq ((x, y), z)$ , but there's a natural isomorphism  $(X \times Y) \times Z \cong X \times (Y \times Z)$ , and this is our associator. Similar considerations give the left and right unitors. In this monoidal category, the tensor product of  $f: X \rightarrow Y$  and  $f': X' \rightarrow Y'$  is the function

$$f \times f' : X \times X' \rightarrow Y \times Y'$$

$$(x, x') \mapsto (f(x), f'(x')).$$

There is also a way to make  $\text{Set}$  into a monoidal category where  $X \otimes Y$  is the disjoint union of  $X$  and  $Y$ , which we shall denote by  $X + Y$ . Here the unit object is the empty set. Again, as indeed with all these examples, the associative law and left/right unit laws hold only up to natural isomorphism. In this monoidal category, the tensor product of  $f: X \rightarrow Y$  and  $f': X' \rightarrow Y'$  is the function

$$f + f' : X + X' \rightarrow Y + Y'$$

$$x \mapsto \begin{cases} f(x) & \text{if } x \in X, \\ f'(x) & \text{if } x \in X'. \end{cases}$$

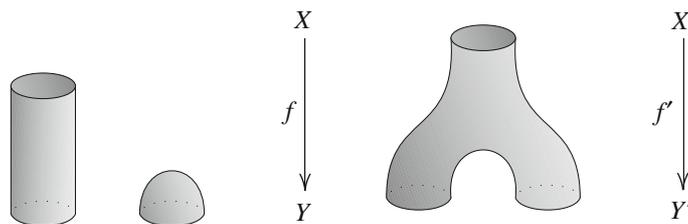
*However, in what follows, when we speak of  $\text{Set}$  as a monoidal category, we always use the cartesian product!*

- There is a way to make  $\text{Hilb}$  into a monoidal category with the usual tensor product of Hilbert spaces:  $\mathbb{C}^n \otimes \mathbb{C}^m \cong \mathbb{C}^{nm}$ . In this case the unit object  $I$  can be taken to be a 1-dimensional Hilbert space, for example  $\mathbb{C}$ .

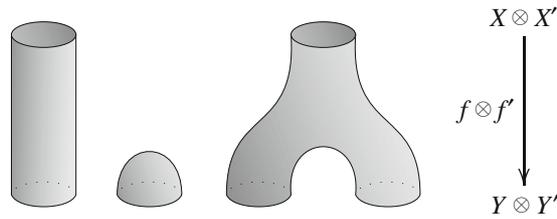
There is also a way to make  $\text{Hilb}$  into a monoidal category where the tensor product is the direct sum:  $\mathbb{C}^n \oplus \mathbb{C}^m \cong \mathbb{C}^{n+m}$ . In this case the unit object is the zero-dimensional Hilbert space,  $\{0\}$ .

*However, in what follows, when we speak of  $\text{Hilb}$  as a monoidal category, we always use the usual tensor product!*

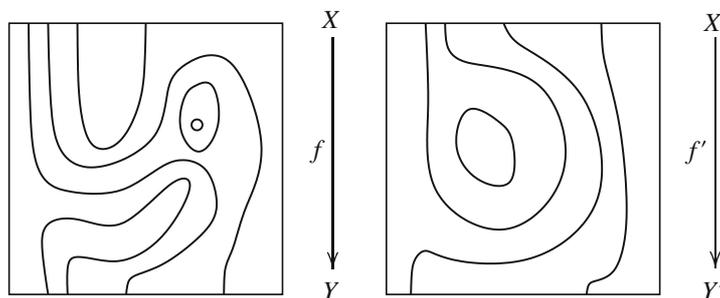
- The tensor product of objects and morphisms in  $n\text{Cob}$  is given by disjoint union. For example, the tensor product of these two morphisms:



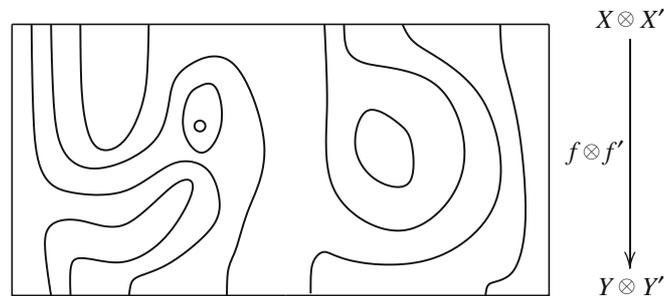
is this:



- The category  $\text{Tang}_k$  is monoidal when  $k \geq 1$ , where the tensor product is given by disjoint union. For example, given these two tangles:



their tensor product looks like this:



The example of  $\text{Set}$  with its cartesian product is different from our other three main examples, because the cartesian product of sets  $X \times X'$  comes equipped with functions called “projections” to the sets  $X$  and  $X'$ :

$$X \xleftarrow{p} X \times X' \xrightarrow{p'} X'$$

Our other main examples lack this feature—though  $\text{Hilb}$  made into a monoidal category using  $\oplus$  has projections. Also, every set has a unique function to the one-element set:

$$!_X: X \rightarrow I.$$

Again, our other main examples lack this feature, though Hilb made into a monoidal category using  $\oplus$  has it. A fascinating feature of quantum mechanics is that we make Hilb into a monoidal category using  $\otimes$  instead of  $\oplus$ , even though the latter approach would lead to a category more like Set.

We can isolate the special features of the cartesian product of sets and its projections, obtaining a definition that applies to any category:

**Definition 8** Given objects  $X$  and  $X'$  in some category, we say an object  $X \times X'$  equipped with morphisms

$$X \xleftarrow{p} X \times X' \xrightarrow{p'} X'$$

is a **cartesian product** (or simply **product**) of  $X$  and  $X'$  if for any object  $Q$  and morphisms

$$\begin{array}{ccc} & Q & \\ f \swarrow & & \searrow f' \\ X & & X' \end{array}$$

there exists a unique morphism  $g: Q \rightarrow X \times X'$  making the following diagram commute:

$$\begin{array}{ccccc} & & Q & & \\ & & \downarrow g & & \\ f \swarrow & & & & \searrow f' \\ X & & X \times X' & & X' \\ p \longleftarrow & & & & \longrightarrow p' \end{array}$$

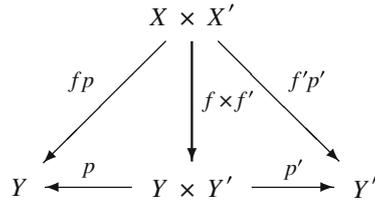
(That is,  $f = pg$  and  $f' = p'g$ .) We say a category has **binary products** if every pair of objects has a product.

The product may not exist, and it may not be unique, but when it exists it is unique up to a canonical isomorphism. This justifies our speaking of “the” product of objects  $X$  and  $Y$  when it exists, and denoting it as  $X \times Y$ .

The definition of cartesian product, while absolutely fundamental, is a bit scary at first sight. To illustrate its power, let us do something with it: combine two morphisms  $f: X \rightarrow Y$  and  $f': X' \rightarrow Y'$  into a single morphism

$$f \times f': X \times X' \rightarrow Y \times Y'.$$

The definition of cartesian product says how to build a morphism of this sort out of a pair of morphisms: namely, morphisms from  $X \times X'$  to  $Y$  and  $Y'$ . If we take these to be  $fp$  and  $f'p'$ , we obtain  $f \times f'$ :



Next, let us isolate the special features of the one-element set:

**Definition 9** An object  $1$  in a category  $C$  is **terminal** if for any object  $Q \in C$  there exists a unique morphism from  $Q$  to  $1$ , which we denote as  $!_Q: Q \rightarrow 1$ .

Again, a terminal object may not exist and may not be unique, but it is unique up to a canonical isomorphism. This is why we can speak of ‘the’ terminal object of a category, and denote it by a specific symbol,  $1$ .

We have introduced the concept of binary products. One can also talk about  $n$ -ary products for other values of  $n$ , but a category with binary products has  $n$ -ary products for all  $n \geq 1$ , since we can construct these as iterated binary products. The case  $n = 1$  is trivial, since the product of one object is just that object itself (up to canonical isomorphism). The remaining case is  $n = 0$ . The zero-ary product of objects, if it exists, is just the terminal object. So, we make the following definition:

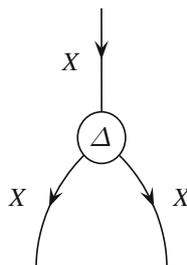
**Definition 10** A category has **finite products** if it has binary products and a terminal object.

A category with finite products can always be made into a monoidal category by choosing a specific product  $X \times Y$  to be the tensor product  $X \otimes Y$ , and choosing a specific terminal object to be the unit object. It takes a bit of work to show this! A monoidal category of this form is called **cartesian**.

In a cartesian category, we can “duplicate and delete information”. In general, the definition of cartesian products gives a way to take two morphisms  $f_1: Q \rightarrow X$  and  $f_2: Q \rightarrow Y$  and combine them into a single morphism from  $Q$  to  $X \times Y$ . If we take  $Q = X = Y$  and take  $f_1$  and  $f_2$  to be the identity, we obtain the **diagonal** or **duplication** morphism:

$$\Delta_X: X \rightarrow X \times X.$$

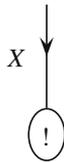
In the category  $\text{Set}$  one can check that this maps any element  $x \in X$  to the pair  $(x, x)$ . In general, we can draw the diagonal as follows:



Similarly, we call the unique map to the terminal object

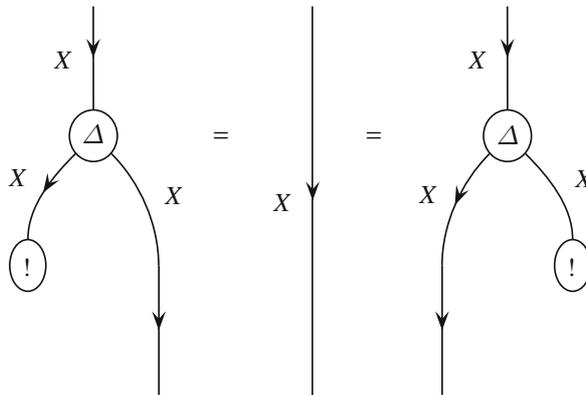
$$!_X : X \rightarrow 1$$

the **deletion** morphism, and draw it as follows:



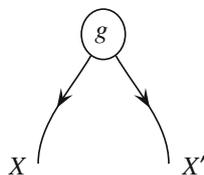
Note that we draw the unit object as an empty space.

A fundamental fact about cartesian categories is that duplicating something and then deleting either copy is the same as doing nothing at all! In string diagrams, this says:

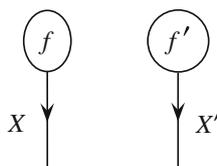


We leave the proof as an exercise for the reader.

Many of the puzzling features of quantum theory come from the noncartesian-ness of the usual tensor product in Hilb. For example, in a cartesian category, every morphism



is actually of the form

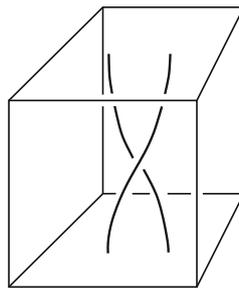


In the case of  $\text{Set}$ , this says that every point of the set  $X \times X'$  comes from a point of  $X$  and a point of  $X'$ . In physics, this would say that every state  $g$  of the combined system  $X \otimes X'$  is built by combining states of the systems  $X$  and  $X'$ . Bell's theorem [20] says that is *not* true in quantum theory. The reason is that quantum theory uses the noncartesian monoidal category  $\text{Hilb}$ !

Also, in quantum theory we *cannot* freely duplicate or delete information. Wootters and Zurek [110] proved a precise theorem to this effect, focused on duplication: the “no-cloning theorem”. One can also prove a “no-deletion theorem”. Again, these results rely on the noncartesian tensor product in  $\text{Hilb}$ .

### 2.2.4 Braided Monoidal Categories

In physics, there is often a process that lets us “switch” two systems by moving them around each other. In topology, there is a tangle that describes the process of switching two points:



In logic, we can switch the order of two statements in a conjunction: the statement “ $X$  and  $Y$ ” is isomorphic to “ $Y$  and  $X$ ”. In computation, there is a simple program that switches the order of two pieces of data. A monoidal category in which we can do this sort of thing is called “braided”:

**Definition 11** A **braided monoidal category** consists of:

- a monoidal category  $C$ ,
- a natural isomorphism called the **braiding** that assigns to every pair of objects  $X, Y \in C$  an isomorphism

$$b_{X,Y}: X \otimes Y \rightarrow Y \otimes X,$$

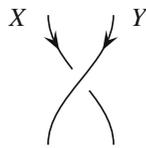
such that the **hexagon equations** hold:

$$\begin{array}{ccccc}
 X \otimes (Y \otimes Z) & \xrightarrow{a_{X,Y,Z}^{-1}} & (X \otimes Y) \otimes Z & \xrightarrow{b_{X,Y} \otimes 1_Z} & (Y \otimes X) \otimes Z \\
 \downarrow b_{X,Y \otimes Z} & & & & \downarrow a_{Y,X,Z} \\
 (Y \otimes Z) \otimes X & \xleftarrow{a_{Y,Z,X}^{-1}} & Y \otimes (Z \otimes X) & \xleftarrow{1_Y \otimes b_{X,Z}} & Y \otimes (X \otimes Z)
 \end{array}$$
  

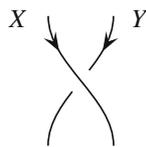
$$\begin{array}{ccccc}
 (X \otimes Y) \otimes Z & \xrightarrow{a_{X,Y,Z}} & X \otimes (Y \otimes Z) & \xrightarrow{1_X \otimes b_{Y,Z}} & X \otimes (Z \otimes Y) \\
 \downarrow b_{X \otimes Y,Z} & & & & \downarrow a_{X,Z,Y}^{-1} \\
 Z \otimes (X \otimes Y) & \xleftarrow{a_{Z,X,Y}} & (Z \otimes X) \otimes Y & \xleftarrow{b_{X,Z} \otimes 1_Y} & (X \otimes Z) \otimes Y
 \end{array}$$

The first hexagon equation says that switching the object  $X$  past  $Y \otimes Z$  all at once is the same as switching it past  $Y$  and then past  $Z$  (with some associators thrown in to move the parentheses). The second one is similar: it says switching  $X \otimes Y$  past  $Z$  all at once is the same as doing it in two steps.

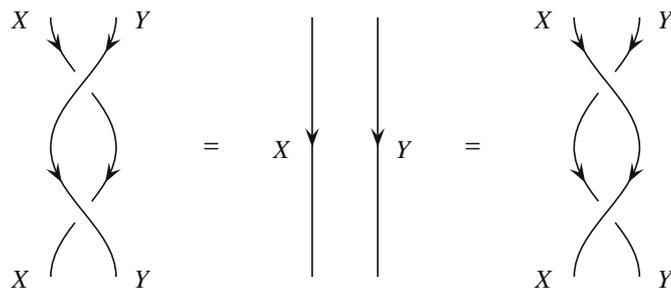
In string diagrams, we draw the braiding  $b_{X,Y} : X \otimes Y \rightarrow Y \otimes X$  like this:



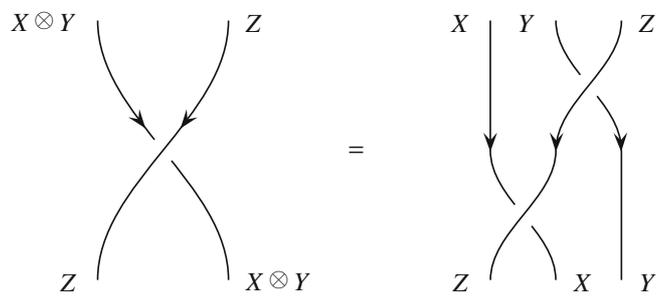
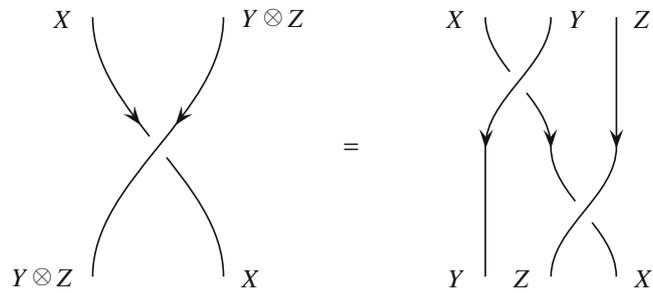
We draw its inverse  $b_{X,Y}^{-1}$  like this:



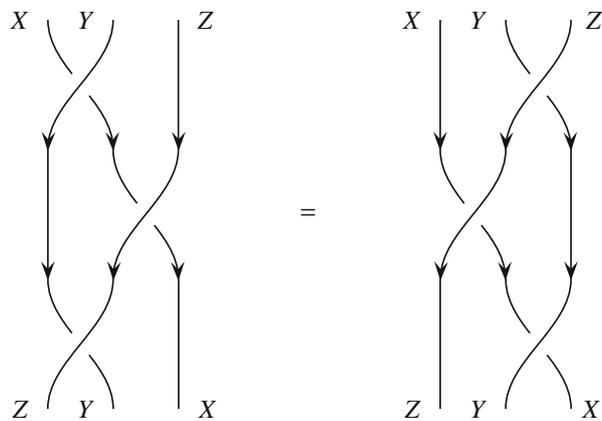
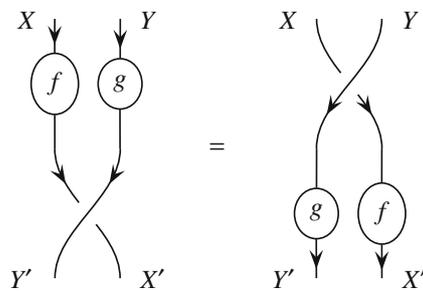
This is a nice notation, because it makes the equations saying that  $b_{X,Y}$  and  $b_{X,Y}^{-1}$  are inverses “topologically true”:



Here are the hexagon equations as string diagrams:



For practice, we urge you to prove the following equations:



If you get stuck, here are some hints. The first equation follows from the naturality of the braiding. The second is called the **Yang–Baxter equation** and follows from a combination of naturality and the hexagon equations [61, 62].

Next, here are some examples. There can be many different ways to give a monoidal category a braiding, or none. However, most of our favorite examples come with well-known “standard” braidings:

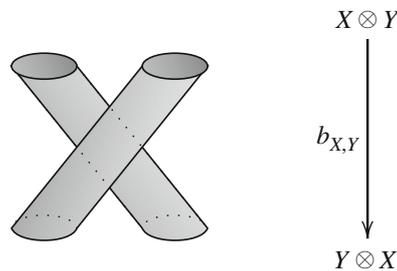
- Any cartesian category automatically becomes braided, and in  $\text{Set}$  with its cartesian product, this standard braiding is given by:

$$b_{X,Y} : X \times Y \rightarrow Y \times X \\ (x, y) \mapsto (y, x).$$

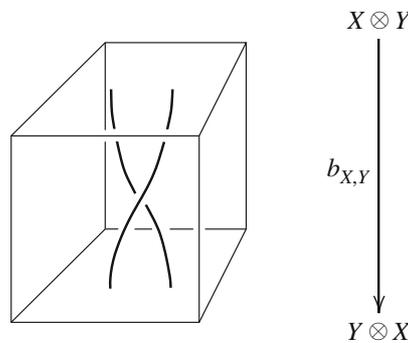
- In  $\text{Hilb}$  with its usual tensor product, the standard braiding is given by:

$$b_{X,Y} : X \otimes Y \rightarrow Y \otimes X \\ x \otimes y \mapsto y \otimes x.$$

- The monoidal category  $n\text{Cob}$  has a standard braiding where  $b_{X,Y}$  is diffeomorphic to the disjoint union of cylinders  $X \times [0, 1]$  and  $Y \times [0, 1]$ . For  $2\text{Cob}$  this braiding looks as follows when  $X$  and  $Y$  are circles:



- The monoidal category  $\text{Tang}_k$  has a standard braiding when  $k \geq 2$ . For  $k = 2$  this looks as follows when  $X$  and  $Y$  are each a single point:



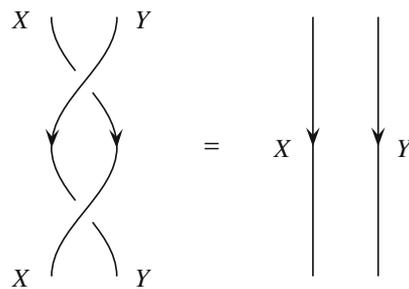
The example of  $\text{Tang}_k$  illustrates an important pattern.  $\text{Tang}_0$  is just a category, because in 0-dimensional space we can only do processes in “series”: that is, compose morphisms.  $\text{Tang}_1$  is a monoidal category, because in 1-dimensional space we can also do processes in “parallel”: that is, tensor morphisms.  $\text{Tang}_2$  is a braided monoidal category, because in 2-dimensional space there is room to move one object around another. Next we shall see what happens when space has 3 or more dimensions!

### 2.2.5 Symmetric Monoidal Categories

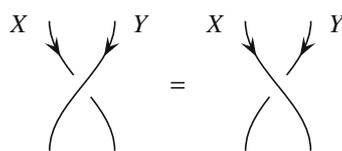
Sometimes switching two objects and switching them again is the same as doing nothing at all. Indeed, this situation is very familiar. So, the first braided monoidal categories to be discovered were “symmetric” ones [77]:

**Definition 12** A **symmetric monoidal category** is a braided monoidal category where the braiding satisfies  $b_{X,Y} = b_{Y,X}^{-1}$ .

So, in a symmetric monoidal category,



or equivalently:



Any cartesian category automatically becomes a symmetric monoidal category, so  $\text{Set}$  is symmetric. It is also easy to check that  $\text{Hilb}$ ,  $n\text{Cob}$  are symmetric monoidal categories. So is  $\text{Tang}_k$  for  $k \geq 3$ .

Interestingly,  $\text{Tang}_k$  “stabilizes” at  $k = 3$ : increasing the value of  $k$  beyond this value merely gives a category equivalent to  $\text{Tang}_3$ . The reason is that we can already untie all knots in 4-dimensional space; adding extra dimensions has no real effect. In fact,  $\text{Tang}_k$  for  $k \geq 3$  is equivalent to  $1\text{Cob}$ . This is part of a conjectured larger pattern called the “Periodic Table” of  $n$ -categories [14]. A piece of this is shown in Table 2.3.

An  $n$ -category has not only morphisms going between objects, but 2-morphisms going between morphisms, 3-morphisms going between 2-morphisms and so on up

**Table 2.3** The periodic table: conjectured descriptions of  $(n + k)$ -categories with only one  $j$ -morphism for  $j < k$ 

	$n = 0$	$n = 1$	$n = 2$
$k = 0$	Sets	Categories	2-Categories
$k = 1$	Monoids	Monoidal categories	Monoidal 2-categories
$k = 2$	Commutative monoids	Braided monoidal categories	Braided monoidal 2-categories
$k = 3$	“	Symmetric monoidal categories	Sylleptic monoidal 2-categories
$k = 4$	“	“	Symmetric monoidal 2-categories
$k = 5$	“	“	“
$k = 6$	“	“	“

to  $n$ -morphisms. In topology we can use  $n$ -categories to describe tangled higher-dimensional surfaces [15], and in physics we can use them to describe not just particles but also strings and higher-dimensional membranes [14, 16]. The Rosetta Stone we are describing concerns only the  $n = 1$  column of the Periodic Table. So, it is probably just a fragment of a larger, still buried  $n$ -categorical Rosetta Stone.

### 2.2.6 Closed Categories

In quantum mechanics, one can encode a linear operator  $f : X \rightarrow Y$  into a quantum state using a technique called “gate teleportation” [51]. In topology, there is a way to take any tangle  $f : X \rightarrow Y$  and bend the input back around to make it part of the output. In logic, we can take a proof that goes from some assumption  $X$  to some conclusion  $Y$  and turn it into a proof that goes from no assumptions to the conclusion “ $X$  implies  $Y$ ”. In computer science, we can take any program that takes input of type  $X$  and produces output of type  $Y$ , and think of it as a piece of data of a new type: a “function type”. The underlying concept that unifies all these examples is the concept of a ‘closed category’.

Given objects  $X$  and  $Y$  in any category  $C$ , there is a *set* of morphisms from  $X$  to  $Y$ , denoted  $\text{hom}(X, Y)$ . In a closed category there is also an *object* of morphisms from  $X$  to  $Y$ , which we denote by  $X \multimap Y$ . (Many other notations are also used.) In this situation we speak of an “internal hom”, since the object  $X \multimap Y$  lives inside  $C$ , instead of “outside”, in the category of sets.

Closed categories were introduced in 1966, by Eilenberg and Kelly [41]. While these authors were able to define a closed structure for any category, it turns out that the internal hom is most easily understood for monoidal categories. The reason is that when our category has a tensor product, it is closed precisely when morphisms from  $X \otimes Y$  to  $Z$  are in natural one-to-one correspondence with morphisms from  $Y$  to  $X \multimap Z$ . In other words, it is closed when we have a natural isomorphism

$$\begin{aligned} \text{hom}(X \otimes Y, Z) &\cong \text{hom}(Y, X \multimap Z) \\ f &\mapsto \tilde{f} \end{aligned}$$

For example, in the category  $\text{Set}$ , if we take  $X \otimes Y$  to be the cartesian product  $X \times Y$ , then  $X \multimap Z$  is just the set of functions from  $X$  to  $Z$ , and we have a one-to-one correspondence between

- functions  $f$  that eat elements of  $X \times Y$  and spit out elements of  $Z$

and

- functions  $\tilde{f}$  that eat elements of  $Y$  and spit out functions from  $X$  to  $Z$ .

This correspondence goes as follows:

$$\tilde{f}(x)(y) = f(x, y).$$

Before considering other examples, we should make the definition of “closed monoidal category” completely precise. For this we must note that for any category  $C$ , there is a functor

$$\text{hom}: C^{\text{op}} \times C \rightarrow \text{Set}.$$

**Definition 13** The **opposite category**  $C^{\text{op}}$  of a category  $C$  has the same objects as  $C$ , but a morphism  $f: x \rightarrow y$  in  $C^{\text{op}}$  is a morphism  $f: y \rightarrow x$  in  $C$ , and the composite  $gf$  in  $C^{\text{op}}$  is the composite  $fg$  in  $C$ .

**Definition 14** For any category  $C$ , the **hom functor**

$$\text{hom}: C^{\text{op}} \times C \rightarrow \text{Set}$$

sends any object  $(X, Y) \in C^{\text{op}} \times C$  to the set  $\text{hom}(X, Y)$ , and sends any morphism  $(f, g) \in C^{\text{op}} \times C$  to the function

$$\begin{aligned} \text{hom}(f, g): \text{hom}(X, Y) &\rightarrow \text{hom}(X', Y') \\ h &\mapsto ghf \end{aligned}$$

when  $f: X' \rightarrow X$  and  $g: Y \rightarrow Y'$  are morphisms in  $C$ .

**Definition 15** A monoidal category  $C$  is **left closed** if there is an **internal hom functor**

$$\multimap: C^{\text{op}} \times C \rightarrow C$$

together with a natural isomorphism  $c$  called **currying** that assigns to any objects  $X, Y, Z \in C$  a bijection

$$c_{X,Y,Z}: \text{hom}(X \otimes Y, Z) \xrightarrow{\sim} \text{hom}(X, Y \multimap Z)$$

It is **right closed** if there is an internal hom functor as above and a natural isomorphism

$$c_{X,Y,Z} : \text{hom}(X \otimes Y, Z) \xrightarrow{\sim} \text{hom}(Y, X \multimap Z).$$

The term “currying” is mainly used in computer science, after the work of Curry [36, 37]. In the rest of this section we only consider *right* closed monoidal categories. Luckily, there is no real difference between left and right closed for a braided monoidal category, as the braiding gives an isomorphism  $X \otimes Y \cong Y \otimes X$ .

All our examples of monoidal categories are closed, but we shall see that, yet again, Set is different from the rest:

- The cartesian category Set is closed, where  $X \multimap Y$  is just the set of functions from  $X$  to  $Y$ . In Set or any other cartesian closed category, the internal hom  $X \multimap Y$  is usually denoted  $Y^X$ . To minimize the number of different notations and emphasize analogies between different contexts, we shall not do this: we shall always use  $X \multimap Y$ . To treat Set as *left* closed, we define the curried version of  $f : X \times Y \rightarrow Z$  as above:

$$\tilde{f}(x)(y) = f(x, y).$$

To treat it as *right* closed, we instead define it by

$$\tilde{f}(y)(x) = f(x, y).$$

This looks a bit awkward, but it will be nice for string diagrams.

- The symmetric monoidal category Hilb with its usual tensor product is closed, where  $X \multimap Y$  is the set of linear operators from  $X$  to  $Y$ , made into a Hilbert space in a standard way. In this case we have an isomorphism

$$X \multimap Y \cong X^* \otimes Y$$

where  $X^*$  is the dual of the Hilbert space  $X$ , that is, the set of linear operators  $f : X \rightarrow \mathbb{C}$ , made into a Hilbert space in the usual way.

- The monoidal category  $\text{Tang}_k$  ( $k \geq 1$ ) is closed. As with Hilb, we have

$$X \multimap Y \cong X^* \otimes Y$$

where  $X^*$  is the orientation-reversed version of  $X$ .

- The symmetric monoidal category  $n\text{Cob}$  is also closed; again

$$X \multimap Y \cong X^* \otimes Y$$

where  $X^*$  is the  $(n - 1)$ -manifold  $X$  with its orientation reversed.

Except for Set, all these examples are actually “compact”. This basically means that  $X \multimap Y$  is isomorphic to  $X^* \otimes Y$ , where  $X^*$  is some object called the “dual”

of  $X$ . But to make this precise, we need to define the ‘dual’ of an object in an arbitrary monoidal category.

To do this, let us generalize from the case of  $\text{Hilb}$ . As already mentioned, each object  $X \in \text{Hilb}$  has a dual  $X^*$  consisting of all linear operators  $f: X \rightarrow I$ , where the unit object  $I$  is just  $\mathbb{C}$ . There is thus a linear operator

$$e_X: X \otimes X^* \rightarrow I \\ x \otimes f \mapsto f(x)$$

called the **counit** of  $X$ . Furthermore, the space of all linear operators from  $X$  to  $Y \in \text{Hilb}$  can be identified with  $X^* \otimes Y$ . So, there is also a linear operator called the **unit** of  $X$ :

$$i_X: I \rightarrow X^* \otimes X \\ c \mapsto c 1_X$$

sending any complex number  $c$  to the corresponding multiple of the identity operator.

The significance of the unit and counit become clearer if we borrow some ideas from Feynman. In physics, if  $X$  is the Hilbert space of internal states of some particle,  $X^*$  is the Hilbert space for the corresponding antiparticle. Feynman realized that it is enlightening to think of antiparticles as particles going backwards in time. So, we draw a wire labelled by  $X^*$  as a wire labelled by  $X$ , but with an arrow pointing ‘backwards in time’: that is, up instead of down:

$$X^* \downarrow = X \uparrow$$

(Here we should admit that most physicists use the opposite convention, where time marches up the page. Since we read from top to bottom, we prefer to let time run down the page.)

If we draw  $X^*$  as  $X$  going backwards in time, we can draw the unit as a **cap**:

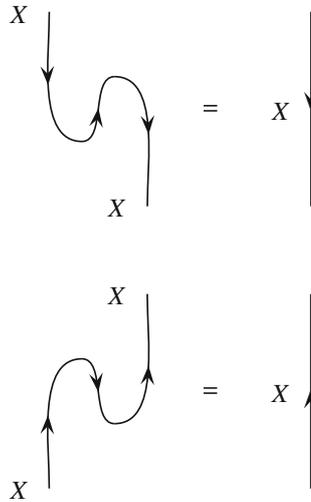
$$X \begin{array}{c} \curvearrowright \\ \uparrow \end{array} X$$

and the counit as a **cup**:

$$X \begin{array}{c} \downarrow \\ \curvearrowleft \end{array} X$$

In Feynman diagrams, these describe the *creation* and *annihilation* of virtual particle-antiparticle pairs!

It then turns out that the unit and counit satisfy two equations, the **zig-zag equations**:



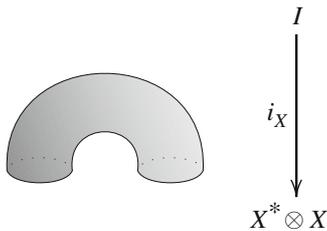
Verifying these is a fun exercise in linear algebra, which we leave to the reader. If we write these equations as commutative diagrams, we need to include some associators and unitors, and they become a bit intimidating:

$$\begin{array}{ccccc}
 X \otimes I & \xrightarrow{1_X \otimes i_X} & X \otimes (X^* \otimes X) & \xrightarrow{a_{X, X^*, X}^{-1}} & (X \otimes X^*) \otimes X \\
 \downarrow r_X & & & & \downarrow e_X \otimes 1_X \\
 X & \xleftarrow{l_X} & I \otimes X & & 
 \end{array}$$

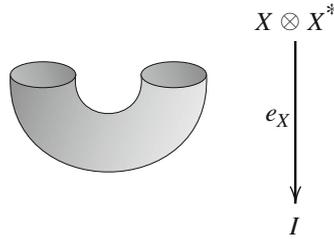
$$\begin{array}{ccccc}
 I \otimes X^* & \xrightarrow{i_X \otimes 1_X} & (X^* \otimes X) \otimes X^* & \xrightarrow{a_{X^*, X, X^*}} & X^* \otimes (X \otimes X^*) \\
 \downarrow l_X & & & & \downarrow 1_{X^*} \otimes e_X \\
 X^* & \xleftarrow{r_{X^*}} & X^* \otimes I & & 
 \end{array}$$

But, they really just say that zig-zags in string diagrams can be straightened out.

This is particularly vivid in examples like  $\text{Tang}_k$  and  $n\text{Cob}$ . For example, in  $2\text{Cob}$ , taking  $X$  to be the circle, the unit looks like this:



while the counit looks like this:



In this case, the zig-zag identities say we can straighten a wiggly piece of pipe.

Now we are ready for some definitions:

**Definition 16** Given objects  $X^*$  and  $X$  in a monoidal category, we call  $X^*$  a **right dual** of  $X$ , and  $X$  a **left dual** of  $X^*$ , if there are morphisms

$$i_X: I \rightarrow X^* \otimes X$$

and

$$e_X: X \otimes X^* \rightarrow I,$$

called the **unit** and **counit** respectively, satisfying the zig-zag equations.

One can show that the left or right dual of an object is unique up to canonical isomorphism. So, we usually speak of “the” right or left dual of an object, when it exists.

**Definition 17** A monoidal category  $C$  is **compact** if every object  $X \in C$  has both a left dual and a right dual.

Often the term “autonomous” is used instead of “compact” here. Many authors reserve the term “compact” for the case where  $C$  is symmetric or at least braided; then left duals are the same as right duals, and things simplify [46]. To add to the confusion, compact symmetric monoidal categories are often called simply “compact closed categories”.

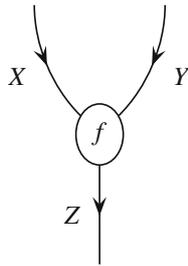
A partial explanation for the last piece of terminology is that any compact monoidal category is automatically closed! For this, we define the internal hom on objects by

$$X \multimap Y = X^* \otimes Y.$$

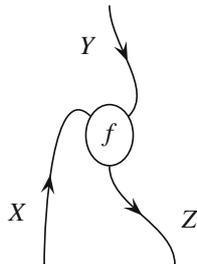
We must then show that the  $*$  operation extends naturally to a functor  $*$ :  $C \rightarrow C$ , so that  $\multimap$  is actually a functor. Finally, we must check that there is a natural isomorphism

$$\text{hom}(X \otimes Y, Z) \cong \text{hom}(Y, X^* \otimes Z)$$

In terms of string diagrams, this isomorphism takes any morphism



and bends back the input wire labelled  $X$  to make it an output:



Now, in a compact monoidal category, we have:

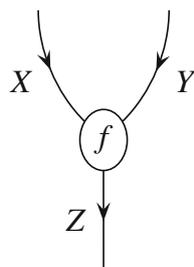
$$\begin{array}{c} \uparrow \\ X \end{array} \quad \begin{array}{c} \downarrow \\ Z \end{array} = \begin{array}{c} \downarrow \\ X \multimap Z \end{array}$$

But in general, closed monoidal categories don't allow arrows pointing up! So for these, drawing the internal hom is more of a challenge. We can use the same style of notation as long as we add a decoration—a **clasp**—that binds two strings together:

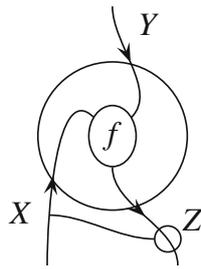
$$\begin{array}{c} \uparrow \\ X \end{array} \quad \begin{array}{c} \downarrow \\ Z \end{array} \quad \text{with a clasp} := \begin{array}{c} \downarrow \\ X \multimap Z \end{array}$$

Only when our closed monoidal category happens to be compact can we eliminate the clasp.

Suppose we are working in a closed monoidal category. Since we draw a morphism  $f : X \otimes Y \rightarrow Z$  like this:



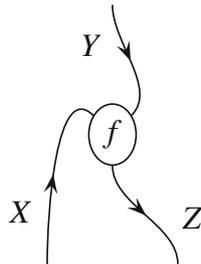
we can draw its curried version  $\tilde{f}: Y \rightarrow X \multimap Z$  by bending down the input wire labelled  $X$  to make it part of the output:



Note that where we bent back the wire labelled  $X$ , a cap like this appeared:



Closed monoidal categories don't really have a cap unless they are compact. So, we drew a **bubble** enclosing  $f$  and the cap, to keep us from doing any illegal manipulations. In the compact case, both the bubble and the clasp are unnecessary, so we can draw  $\tilde{f}$  like this:



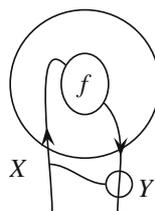
An important special case of currying gives the **name** of a morphism  $f: X \rightarrow Y$ ,

$$\ulcorner f \urcorner: I \rightarrow X \multimap Y.$$

This is obtained by currying the morphism

$$f r_x: I \otimes X \rightarrow Y.$$

In string diagrams, we draw  $\ulcorner f \urcorner$  as follows:



In the category  $\text{Set}$ , the unit object is the one-element set,  $1$ . So, a morphism from this object to a set  $Q$  picks out a point of  $Q$ . In particular, the name  $\lceil f \rceil: 1 \rightarrow X \multimap Y$  picks out the element of  $X \multimap Y$  corresponding to the function  $f: X \rightarrow Y$ . More generally, in any cartesian closed category the unit object is the terminal object  $1$ , and a morphism from  $1$  to an object  $Q$  is called a **point** of  $Q$ . So, even in this case, we can say the name of a morphism  $f: X \rightarrow Y$  is a point of  $X \multimap Y$ .

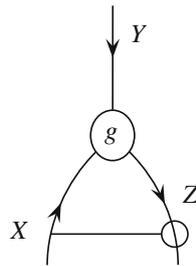
Something similar works for  $\text{Hilb}$ , though this example is compact rather than cartesian. In  $\text{Hilb}$ , the unit object  $I$  is just  $\mathbb{C}$ . So, a nonzero morphism from  $I$  to any Hilbert space  $Q$  picks out a nonzero vector in  $Q$ , which we can normalize to obtain a **state** in  $Q$ : that is, a unit vector. In particular, the name of a nonzero morphism  $f: X \rightarrow Y$  gives a state of  $X^* \otimes Y$ . This method of encoding operators as states is the basis of “gate teleportation” [51].

Currying is a bijection, so we can also **uncurry**:

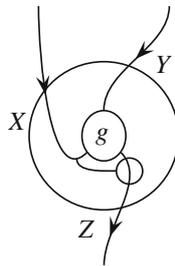
$$c_{X,Y,Z}^{-1}: \text{hom}(Y, X \multimap Z) \xrightarrow{\sim} \text{hom}(X \otimes Y, Z)$$

$$g \mapsto \hat{g}.$$

Since we draw a morphism  $g: Y \rightarrow X \multimap Z$  like this:



we draw its “uncurried” version  $\hat{g}: X \otimes Y \rightarrow Z$  by bending the output  $X$  up to become an input:



Again, we must put a bubble around the “cup” formed when we bend down the wire labelled  $Y$ , unless we are in a compact monoidal category.

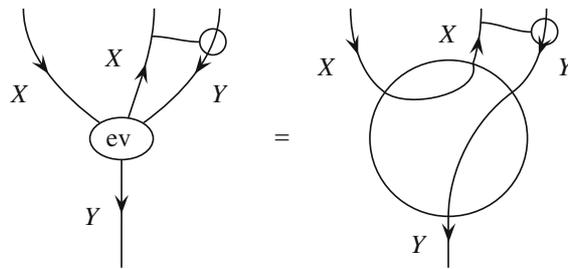
A good example of uncurrying is the **evaluation** morphism:

$$\text{ev}_{X,Y}: X \otimes (X \multimap Y) \rightarrow Y.$$

This is obtained by uncurrying the identity

$$1_{X \multimap Y} : (X \multimap Y) \rightarrow (X \multimap Y).$$

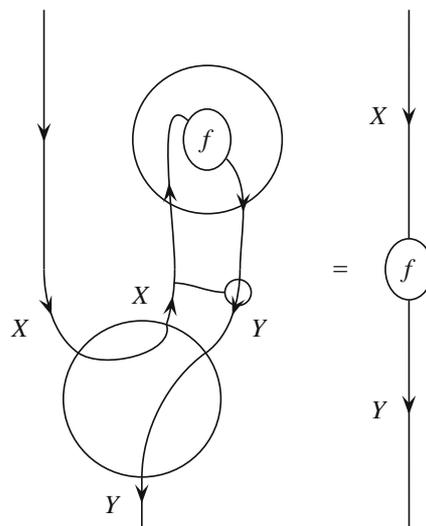
In  $\text{Set}$ ,  $\text{ev}_{X,Y}$  takes any function from  $X$  to  $Y$  and evaluates it at any element of  $X$  to give an element of  $Y$ . In terms of string diagrams, the evaluation morphism looks like this:



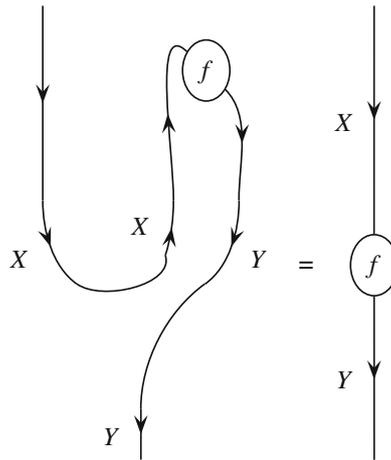
In any closed monoidal category, we can recover a morphism from its name using evaluation. More precisely, this diagram commutes:

$$\begin{array}{ccc}
 X \otimes I & \xleftarrow{r^{-1}} & X \\
 \downarrow 1_{X \otimes f} & & \downarrow f \\
 X \otimes (X \multimap Y) & \xrightarrow{\text{ev}_{X,Y}} & Y
 \end{array}$$

Or, in terms of string diagrams:



We leave the proof of this as an exercise. In general, one must use the naturality of currying. In the special case of a compact monoidal category, there is a nice picture proof! Simply pop the bubbles and remove the clasp:



The result then follows from one of the zig-zag identities.

In our rapid introduction to string diagrams, we have not had time to illustrate how these diagrams become a powerful tool for solving concrete problems. So, here are some starting points for further study:

- Representations of Lie groups play a fundamental role in quantum physics, especially gauge field theory. Every Lie group has a compact symmetric monoidal category of finite-dimensional representations. In his book *Group Theory*, Cvitanovic [38] develops detailed string diagram descriptions of these representation categories for the classical Lie groups  $SU(n)$ ,  $SO(n)$ ,  $SU(n)$  and also the more exotic “exceptional” Lie groups. His book also illustrates how this technology can be used to simplify difficult calculations in gauge field theory.
- Quantum groups are a generalization of groups which show up in 2d and 3d physics. The big difference is that a quantum group has compact *braided* monoidal category of finite-dimensional representations. Kauffman’s *Knots and Physics* [65] is an excellent introduction to how quantum groups show up in knot theory and physics; it is packed with string diagrams. For more details on quantum groups and braided monoidal categories, see the book by Kassel [64].
- Kauffman and Lins [66] have written a beautiful string diagram treatment of the category of representations of the simplest quantum group,  $SU_q(2)$ . They also use it to construct some famous 3-manifold invariants associated to 3d and 4d topological quantum field theories: the Witten–Reshetikhin–Turaev, Turaev–Viro and Crane–Yetter invariants. In this example, string diagrams are often called “ $q$ -deformed spin networks” [101]. For generalizations to other quantum groups, see the more advanced texts by Turaev [107] and by Bakalov and Kirillov [17]. The key ingredient is a special class of compact braided monoidal categories called “modular tensor categories”.
- Kock [70] has written a nice introduction to 2d topological quantum field theories which uses diagrammatic methods to work with 2Cob.
- Abramsky, Coecke and collaborators [2–4, 31, 33, 34] have developed string diagrams as a tool for understanding quantum computation. The easiest introduction is Coecke’s “Kindergarten quantum mechanics” [32].

### 2.2.7 Dagger Categories

Our discussion would be sadly incomplete without an important admission: *nothing we have done so far with Hilbert spaces used the inner product!* So, we have not yet touched on the essence of quantum theory.

Everything we have said about Hilb applies equally well to Vect: the category of finite-dimensional *vector spaces* and linear operators. Both Hilb and Vect are compact symmetric monoidal categories. In fact, these compact symmetric monoidal categories are “equivalent” in a certain precise sense [78].

So, what makes Hilb different? In terms of category theory, the special thing is that we can take the Hilbert space adjoint of any linear operator  $f: X \rightarrow Y$  between finite-dimensional Hilbert spaces, getting an operator  $f^\dagger: Y \rightarrow X$ . This ability to ‘reverse’ morphisms makes Hilb into a ‘dagger category’:

**Definition 18** A **dagger category** is a category  $C$  such that for any morphism  $f: X \rightarrow Y$  in  $C$  there is a specified morphism  $f^\dagger: Y \rightarrow X$  such that

$$(gf)^\dagger = f^\dagger g^\dagger$$

for every pair of composable morphisms  $f$  and  $g$ , and

$$(f^\dagger)^\dagger = f$$

for every morphism  $f$ .

Equivalently, a dagger category is one equipped with a functor  $\dagger: C \rightarrow C^{\text{op}}$  that is the identity on objects and satisfies  $(f^\dagger)^\dagger = f$  for every morphism.

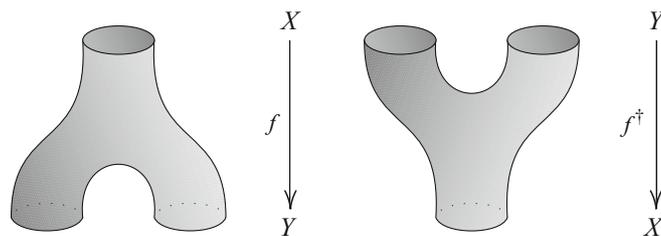
In fact, all our favorite examples of categories can be made into dagger categories, except for Set:

- There is no way to make Set into a dagger category, since there is a function from the empty set to the 1-element set, but none the other way around.
- The category Hilb becomes a dagger category as follows. Given any morphism  $f: X \rightarrow Y$  in Hilb, there is a morphism  $f^\dagger: Y \rightarrow X$ , the **Hilbert space adjoint** of  $f$ , defined by

$$\langle f^\dagger \psi, \phi \rangle = \langle \psi, f\phi \rangle$$

for all  $\phi \in X$ ,  $\psi \in Y$ .

- For any  $k$ , the category  $\text{Tang}_k$  becomes a dagger category where we obtain  $f^\dagger: Y \rightarrow X$  by reflecting  $f: X \rightarrow Y$  in the vertical direction, and then switching the direction of the little arrows denoting the orientations of arcs and circles.
- For any  $n$ , the category  $n\text{Cob}$  becomes a dagger category where we obtain  $f^\dagger: Y \rightarrow X$  by switching the input and output of  $f: X \rightarrow Y$ , and then switching the orientation of each connected component of  $f$ . Again, a picture speaks a thousand words:



In applications to physics, this dagger operation amounts to “switching the future and the past”.

In all the dagger categories above, the dagger structure interacts in a nice way with the monoidal structure and also, when it exists, the braiding. One can write a list of axioms characterizing how this works [2, 3, 97]. So, it seems that the ability to “reverse” morphisms is another way in which categories of a quantum flavor differ from the category of sets and functions. This has important implications for the foundations of quantum theory [12] and also for topological quantum field theory [14], where dagger categories seem to be part of larger story involving “ $n$ -categories with duals” [15]. However, this story is still poorly understood—there is much more work to be done.

## 2.3 Logic

### 2.3.1 Background

Symmetric monoidal closed categories show up not only in physics and topology, but also in logic. We would like to explain how. To set the stage, it seems worthwhile to sketch a few ideas from twentieth-century logic.

Modern logicians study many systems of reasoning beside ordinary classical logic. Of course, even classical logic comes in various degrees of strength. First there is the “propositional calculus”, which allows us to reason with abstract propositions  $X, Y, Z, \dots$  and these logical connectives:

and	$\wedge$
or	$\vee$
implies	$\Rightarrow$
not	$\neg$
true	$\top$
false	$\perp$

Then there is the “predicate calculus”, which also allows variables like  $x, y, z, \dots$ , predicates like  $P(x)$  and  $Q(x, y, z)$ , and the symbols “for all” ( $\forall$ ) and “there exists” ( $\exists$ ), which allow us to quantify over variables. There are also higher-order systems that allow us to quantify over predicates, and so on. To keep things simple, we

mainly confine ourselves to the propositional calculus in what follows. But even here, there are many alternatives to the “classical” version!

The most-studied of these alternative systems are *weaker* than classical logic: they make it harder or even impossible to prove things we normally take for granted. One reason is that some logicians deny that certain familiar principles are actually valid. But there are also subtler reasons. One is that studying systems with rules of lesser strength allows for a fine-grained study of precisely which methods of reasoning are needed to prove which results. Another reason—the one that concerns us most here—is that dropping familiar rules and then adding them back in one at a time sheds light on the connection between logic and category theory.

For example, around 1907 Brouwer [53] began advocating “intuitionism”. As part of this, he raised doubts about the law of excluded middle, which amounts to a rule saying that from  $\neg\neg X$  we can deduce  $X$ . One problem with this principle is that proofs using it are not “constructive”. For example, we may prove by contradiction that some equation has a solution, but still have no clue how to construct the solution. For Brouwer, this meant the principle was invalid.

Anyone who feels the law of excluded middle is invalid is duty-bound to study intuitionistic logic. But, there is another reason for studying this system. Namely: we do not really *lose* anything by dropping the law of excluded middle! Instead, we *gain* a fine-grained distinction: the distinction between a direct proof of  $X$  and a proof by contradiction, which yields merely  $\neg\neg X$ . If we do not care about this distinction we are free to ignore it, but there is no harm in having it around.

In the 1930’s, this idea was made precise by Gödel [49] and Gentzen [104]. They showed that we can embed classical logic in intuitionistic logic. In fact, they found a map sending any formula  $X$  of the propositional calculus to a new formula  $X^\circ$ , such that  $X$  is provable classically if and only if  $X^\circ$  is provable intuitionistically. (More impressively, this map also works for the predicate calculus.)

Later, yet another reason for being interested in intuitionistic logic became apparent: its connection to category theory. In its very simplest form, this connection works as follows. Suppose we have a set of propositions  $X, Y, Z, \dots$  obeying the laws of the intuitionistic propositional calculus. We can create a category  $C$  where these propositions are objects and there is at most one morphism from any object  $X$  to any object  $Y$ : a single morphism when  $X$  implies  $Y$ , and none otherwise!

A category with at most one morphism from any object to any other is called a **preorder**. In the propositional calculus, we often treat two propositions as equal when they both imply each other. If we do this, we get a special sort of preorder: one where isomorphic objects are automatically equal. This special sort of preorder is called a **partially ordered set**, or **poset** for short. Posets abound in logic, precisely because they offer a simple framework for understanding implication.

If we start from a set of propositions obeying the intuitionistic propositional calculus, the resulting category  $C$  is better than a mere poset. It is also cartesian, with  $X \wedge Y$  as the product of  $X$  and  $Y$ , and  $\top$  as the terminal object! To see this, note that any proposition  $Q$  has a unique morphism to  $X \wedge Y$  whenever it has morphisms to  $X$  and to  $Y$ . This is simply a fancy way of saying that  $Q$  implies  $X \wedge Y$  when it implies  $X$  and implies  $Y$ . It is also easy to see that  $\top$  is terminal: anything implies the truth.

Even better, the category  $C$  is cartesian closed, with  $X \Rightarrow Y$  as the internal hom. The reason is that

$$X \wedge Y \text{ implies } Z \quad \text{iff} \quad Y \text{ implies } X \Rightarrow Z.$$

This automatically yields the basic property of the internal hom:

$$\text{hom}(X \otimes Y, Z) \cong \text{hom}(Y, X \multimap Z).$$

Indeed, if the reader is puzzled by the difference between “ $X$  implies  $Y$ ” and  $X \Rightarrow Y$ , we can now explain this more clearly: the former involves the homset  $\text{hom}(X, Y)$  (which has one element when  $X$  implies  $Y$  and none otherwise), while the latter is the internal hom, an object in  $C$ .

So,  $C$  is a cartesian closed poset. But, it also has one more nice property, thanks to the presence of  $\vee$  and  $\perp$ . We have seen that  $\wedge$  and  $\top$  make the category  $C$  cartesian;  $\vee$  and  $\perp$  satisfy exactly analogous rules, but with the implications turned around, so they make  $C^{\text{op}}$  cartesian.

And that is all! In particular, negation gives nothing more, since we can define  $\neg X$  to be  $X \Rightarrow \perp$ , and all its intuitionistically valid properties then follow. So, the kind of category we get from the intuitionistic propositional calculus by taking propositions as objects and implications as morphisms is precisely a **Heyting algebra**: a cartesian closed poset  $C$  such that  $C^{\text{op}}$  is also cartesian.

Heyting, a student of Brouwer, introduced Heyting algebras in intuitionistic logic before categories were even invented. So, he used very different language to define them. But, the category-theoretic approach to Heyting algebras illustrates the connection between cartesian closed categories and logic. It also gives more evidence that dropping the law of excluded middle is an interesting thing to try.

Since we have explained the basics of cartesian closed categories, but not said what happens when the *opposite* of such a category is *also* cartesian, in the sections to come we will take a drastic step and limit our discussion of logic even further. We will neglect  $\vee$  and  $\perp$ , and concentrate only on the fragment of the propositional calculus involving  $\wedge$ ,  $\top$  and  $\Rightarrow$ .

Even here, it turns out, there are interesting things to say—and interesting ways to modify the usual rules. This will be the main subject of the sections to come. But to set the stage, we need to say a bit about proof theory.

Proof theory is the branch of mathematical logic that treats proofs as mathematical entities worthy of study in their own right. It lets us dig deeper into the propositional calculus by studying not merely *whether or not* some assumption  $X$  implies some conclusion  $Y$ , but the whole *set of proofs* leading from  $X$  to  $Y$ . This amounts to studying not just posets (or preorders), but categories that allow many morphisms from one object to another.

In Hilbert’s approach to proof, there were many axioms and just one rule to deduce new theorems: *modus ponens*, which says that from  $X$  and “ $X$  implies  $Y$ ” we can deduce  $Y$ . Most of modern proof theory focuses on another approach, the

“sequent calculus”, due to Gentzen [104]. In this approach there are few axioms but many inference rules.

An excellent introduction to the sequent calculus is the book *Proofs and Types* by Girard, Lafont and Taylor, freely available online [48]. Here we shall content ourselves with some sketchy remarks. A “sequent” is something like this:

$$X_1, \dots, X_m \vdash Y_1, \dots, Y_n$$

where  $X_i$  and  $Y_i$  are propositions. We read this sequent as saying that *all* the propositions  $X_i$ , taken together, can be used to prove at least *one* of the propositions  $Y_i$ . This strange-sounding convention gives the sequent calculus a nice symmetry, as we shall soon see.

In the sequent calculus, an “inference rule” is something that produces new sequents from old. For example, here is the **left weakening** rule:

$$\frac{X_1, \dots, X_m \vdash Y_1, \dots, Y_n}{X_1, \dots, X_m, A \vdash Y_1, \dots, Y_n}$$

This says that from the sequent above the line we can get the sequent below the line: we can throw in the extra assumption  $A$  without harm. Thanks to the strange-sounding convention we mentioned, this rule has a mirror-image version called **right weakening**:

$$\frac{X_1, \dots, X_m \vdash Y_1, \dots, Y_n}{X_1, \dots, X_m \vdash Y_1, \dots, Y_n, A}$$

In fact, Gentzen’s whole setup has this mirror symmetry! For example, his rule called **left contraction**:

$$\frac{X_1, \dots, X_m, A, A \vdash Y_1, \dots, Y_n}{X_1, \dots, X_m, A \vdash Y_1, \dots, Y_n}$$

has a mirror partner called **right contraction**:

$$\frac{X_1, \dots, X_m \vdash Y_1, \dots, Y_n, A, A}{X_1, \dots, X_m \vdash Y_1, \dots, Y_n, A}$$

Similarly, this rule for “and”

$$\frac{X_1, \dots, X_m, A \vdash Y_1, \dots, Y_n}{X_1, \dots, X_m, A \wedge B \vdash Y_1, \dots, Y_n}$$

has a mirror partner for “or”:

$$\frac{X_1, \dots, X_m \vdash Y_1, \dots, Y_n, A}{X_1, \dots, X_m \vdash Y_1, \dots, Y_n, A \vee B}$$

Logicians now realize that this mirror symmetry can be understood in terms of the duality between a category and its opposite.

Gentzen used sequents to write inference rules for the classical propositional calculus, and also the classical predicate calculus. Now, in these forms of logic we have

$$X_1, \dots, X_m \vdash Y_1, \dots, Y_n$$

if and only if we have

$$X_1 \wedge \dots \wedge X_m \vdash Y_1 \vee \dots \vee Y_n.$$

So, why did Gentzen use sequents with a *list* of propositions on each side of the  $\vdash$  symbol, instead just a single proposition? The reason is that this let him use only inference rules having the “subformula property”. This says that every proposition in the sequent above the line appears as part of some proposition in the sequent below the line. So, a proof built from such inference rules becomes a “tree” where all the propositions further up the tree are subformulas of those below.

This idea has powerful consequences. For example, in 1936 Gentzen was able prove the consistency of Peano’s axioms of arithmetic! His proof essentially used induction on trees (Readers familiar with Gödel’s second incompleteness theorem should be reassured that this sort of induction cannot itself be carried out in Peano arithmetic.)

The most famous rule *lacking* the subformula property is the ‘cut rule’:

$$\frac{X_1, \dots, X_m \vdash Y_1, \dots, Y_k, A \quad X_{m+1}, \dots, X_n, A \vdash Y_{k+1}, \dots, Y_\ell}{X_1, \dots, X_n \vdash Y_1, \dots, Y_\ell}$$

From the two sequents on top, the cut rule gives us the sequent below. Note that the intermediate step  $A$  does not appear in the sequent below. It is “cut out”. So, the cut rule lacks the subformula property. But, one of Gentzen’s great achievements was to show that any proof in the classical propositional (or even predicate) calculus that can be done *with* the cut rule can also be done *without* it. This is called ‘cut elimination’.

Gentzen also wrote down inference rules suitable for the intuitionistic propositional and predicate calculi. These rules lack the mirror symmetry of the classical case. But in the 1980s, this symmetry was restored by Girard’s invention of “linear logic” [47].

Linear logic lets us keep track of how many times we use a given premise to reach a given conclusion. To accomplish this, Girard introduced some new logical connectives! For starters, he introduced ‘linear’ connectives called  $\otimes$  and  $\multimap$ , and a logical constant called  $I$ . These act a bit like  $\wedge$ ,  $\Rightarrow$  and  $\top$ . However, they satisfy rules corresponding to a symmetric monoidal category instead of a cartesian closed category. In particular, from  $X$  we can prove neither  $X \otimes X$  nor  $I$ . So, we cannot freely “duplicate” and “delete” propositions using these new connectives. This is

reflected in the fact that linear logic drops Gentzen’s contraction and weakening rules.

By itself, this might seem unbearably restrictive. However, Girard *also* kept the connectives  $\wedge$ ,  $\Rightarrow$  and  $\top$  in his system, still satisfying the usual rules. And, he introduced an operation called the “exponential”,  $!$ , which takes a proposition  $X$  and turns it into an “arbitrary stock of copies of  $X$ ”. So, for example, from  $!X$  we can prove  $1$ , and  $X$ , and  $X \otimes X$ , and  $X \otimes X \otimes X$ , and so on.

Full-fledged linear logic has even more connectives than we have described here. It seems baroque and peculiar at first glance. It also comes in both classical and intuitionistic versions! But, *just as classical logic can be embedded in intuitionistic logic, intuitionistic logic can be embedded in intuitionistic linear logic* [47]. So, we do not lose any deductive power. Instead, we gain the ability to make even more fine-grained distinctions.

In what follows, we discuss the fragment of intuitionistic linear logic involving only  $\otimes$ ,  $-\circ$  and  $I$ . This is called “multiplicative intuitionistic linear logic” [52, 91]. It turns out to be the system of logic suitable for closed symmetric monoidal categories—nothing more or less.

### 2.3.2 Proofs as Morphisms

In Sect. 2.2 we described categories with various amounts of extra structure, starting from categories pure and simple, and working our way up to monoidal categories, braided monoidal categories, symmetric monoidal categories, and so on. Our treatment only scratched the surface of an enormously rich taxonomy. In fact, each kind of category with extra structure corresponds to a system of logic with its own inference rules!

To see this, we will think of *propositions* as *objects* in some category, and *proofs* as giving *morphisms*. Suppose  $X$  and  $Y$  are propositions. Then, we can think of a proof starting from the assumption  $X$  and leading to the conclusion  $Y$  as giving a morphism  $f: X \rightarrow Y$ . (In Sect. 2.3.3 we shall see that a morphism is actually an equivalence class of proofs—but for now let us gloss over this issue.)

Let us write  $X \vdash Y$  when, starting from the assumption  $X$ , there is a proof leading to the conclusion  $Y$ . An inference rule is a way to get new proofs from old. For example, in almost every system of logic, if there is a proof leading from  $X$  to  $Y$ , and a proof leading from  $Y$  to  $Z$ , then there is a proof leading from  $X$  to  $Z$ . We write this inference rule as follows:

$$\frac{X \vdash Y \quad Y \vdash Z}{X \vdash Z}$$

We can call this **cut rule**, since it lets us “cut out” the intermediate step  $Y$ . It is a special case of Gentzen’s cut rule, mentioned in the previous section. It should remind us of composition of morphisms in a category: if we have a morphism  $f: X \rightarrow Y$  and a morphism  $g: Y \rightarrow Z$ , we get a morphism  $gf: X \rightarrow Z$ .

Also, in almost every system of logic there is a proof leading from  $X$  to  $X$ . We can write this as an inference rule that starts with *nothing* and concludes the existence of a proof of  $X$  from  $X$ :

$$\frac{}{X \vdash X}$$

This rule should remind us of how every object in category has an identity morphism: for any object  $X$ , we automatically get a morphism  $1_X: X \rightarrow X$ . Indeed, this rule is sometimes called the **identity rule**.

If we pursue this line of thought, we can take the definition of a closed symmetric monoidal category and extract a collection of inference rules. Each rule is a way to get new morphisms from old in a closed symmetric monoidal category. There are various superficially different but ultimately equivalent ways to list these rules. Here is one:

$$\begin{array}{cc} \frac{}{X \vdash X} \text{ (i)} & \frac{X \vdash Y \quad Y \vdash Z}{X \vdash Z} \text{ (o)} \\ \frac{W \vdash X \quad Y \vdash Z}{W \otimes Y \vdash X \otimes Z} \text{ (\otimes)} & \frac{W \vdash (X \otimes Y) \otimes Z}{W \vdash X \otimes (Y \otimes Z)} \text{ (a)} \\ \frac{X \vdash I \otimes Y}{X \vdash Y} \text{ (l)} & \frac{X \vdash Y \otimes I}{X \vdash Y} \text{ (r)} \\ \frac{W \vdash X \otimes Y}{W \vdash Y \otimes X} \text{ (b)} & \frac{X \otimes Y \vdash Z}{Y \vdash X \multimap Z} \text{ (c)} \end{array}$$

Double lines mean that the inverse rule also holds. We have given each rule a name, written to the right in parentheses. As already explained, rules (i) and (o) come from the presence of identity morphisms and composition in any category. Rules (\otimes), (a), (l), and (r) come from tensoring, the associator, and the left and right unitors in a monoidal category. Rule (b) comes from the braiding in a braided monoidal category, and rule (c) comes from currying in a closed monoidal category.

Now for the big question: *what does all this mean in terms of logic?* These rules describe a small fragment of the propositional calculus. To see this, we should read the connective  $\otimes$  as “and”, the connective  $\multimap$  as “implies”, and the proposition  $I$  as “true”.

In this interpretation, rule (c) says we can turn a proof leading from the assumption “ $Y$  and  $X$ ” to the conclusion  $Z$  into a proof leading from  $X$  to “ $Y$  implies  $Z$ ”. It also says we can do the reverse. This is true in classical, intuitionistic and linear logic, and so are all the other rules. Rules (a) and (b) say that “and” is associative and commutative. Rule (l) says that any proof leading from the assumption  $X$  to the conclusion “true and  $Y$ ” can be converted to a proof leading from  $X$  to  $Y$ , and vice versa. Rule (r) is similar.

What do we do with these rules? We use them to build “deductions”. Here is an easy example:

$$\frac{\overline{X \multimap Y \vdash X \multimap Y}^{(i)}}{X \otimes (X \multimap Y) \vdash Y}^{(c^{-1})}$$

First we use the identity rule, and then the inverse of the currying rule. At the end, we obtain

$$X \otimes (X \multimap Y) \vdash Y.$$

This should remind us of the evaluation morphisms we have in a closed monoidal category:

$$\text{ev}_{X,Y}: X \otimes (X \multimap Y) \rightarrow Y.$$

In terms of logic, the point is that we can prove  $Y$  from  $X$  and “ $X$  implies  $Y$ ”. This fact comes in handy so often that we may wish to abbreviate the above deduction as an extra inference rule—a rule derived from our basic list:

$$\overline{X \otimes (X \multimap Y) \vdash Y}^{(ev)}$$

This rule is called **modus ponens**.

In general, a deduction is a tree built from inference rules. Branches arise when we use the  $(\circ)$  or  $(\otimes)$  rules. Here is an example:

$$\frac{\frac{\overline{(A \otimes B) \otimes C \vdash (A \otimes B) \otimes C}^{(i)}}{\overline{(A \otimes B) \otimes C \vdash A \otimes (B \otimes C)}^{(a)}} \quad A \otimes (B \otimes C) \vdash D}{(A \otimes B) \otimes C \vdash D}^{(\circ)}$$

Again we can abbreviate this deduction as a derived rule. In fact, this rule is reversible:

$$\frac{A \otimes (B \otimes C) \vdash D}{(A \otimes B) \otimes C \vdash D}^{(\omega)}$$

For a more substantial example, suppose we want to show

$$(X \multimap Y) \otimes (Y \multimap Z) \vdash X \multimap Z.$$

The deduction leading to this will not even fit on the page unless we use our abbreviations:

$$\frac{\overline{X \otimes (X \multimap Y) \vdash Y}^{(ev)} \quad \overline{Y \multimap Z \vdash Y \multimap Z}^{(i)}}{\overline{(X \otimes (X \multimap Y)) \otimes (Y \multimap Z) \vdash Y \otimes (Y \multimap Z)}^{(\otimes)} \quad \overline{Y \otimes (Y \multimap Z) \vdash Z}^{(ev)}}{\frac{\overline{(X \otimes (X \multimap Y)) \otimes (Y \multimap Z) \vdash Z}^{(\alpha^{-1})}}{X \otimes ((X \multimap Y) \otimes (Y \multimap Z)) \vdash Z}^{(c)} \quad \overline{(X \multimap Y) \otimes (Y \multimap Z) \vdash X \multimap Z}^{(c)}}$$

Since each of the rules used in this deduction came from a way to get new morphisms from old in a closed monoidal category (we never used the braiding), it follows that in every such category we have **internal composition** morphisms:

$$\bullet_{X,Y,Z} : (X \multimap Y) \otimes (Y \multimap Z) \rightarrow X \multimap Z.$$

These play the same role for the internal hom that ordinary composition

$$\circ : \text{hom}(X, Y) \times \text{hom}(Y, Z) \rightarrow \text{hom}(X, Z)$$

plays for the ordinary hom.

We can go ahead making further deductions in this system of logic, but the really interesting thing is what it omits. For starters, it omits the connective “or” and the proposition “false”. It also omits two inference rules we normally take for granted—namely, **contraction**:

$$\frac{X \vdash Y}{X \vdash Y \otimes Y} \quad (\Delta)$$

and **weakening**:

$$\frac{X \vdash Y}{X \vdash I} \quad (!)$$

which are closely related to duplication and deletion in a cartesian category. Omitting these rules is a distinctive feature of linear logic [47]. The word “linear” should remind us of the category  $\text{Hilb}$ . As noted in Sect. 2.2.3, this category with its usual tensor product is noncartesian, so it does not permit duplication and deletion. But, what does omitting these rules mean *in terms of logic*?

Ordinary logic deals with propositions, so we have been thinking of the above system of logic in the same way. Linear logic deals not just with propositions, but also other resources—for example, physical things! Unlike propositions in ordinary logic, we typically can’t duplicate or delete these other resources. In classical logic, if we know that a proposition  $X$  is true, we can use  $X$  as many or as few times as we like when trying to prove some proposition  $Y$ . But if we have a cup of milk, we can’t use it to make cake and then use it again to make butter. Nor can we make it disappear without a trace: even if we pour it down the drain, it must go somewhere.

In fact, these ideas are familiar in chemistry. Consider the following resources:

$$\begin{aligned} H_2 &= \text{one molecule of hydrogen} \\ O_2 &= \text{one molecule of oxygen} \\ H_2O &= \text{one molecule of water} \end{aligned}$$

We can burn hydrogen, combining one molecule of oxygen with two of hydrogen to obtain two molecules of water. A category theorist might describe this reaction as a morphism:

$$f: O_2 \otimes (H_2 \otimes H_2) \rightarrow H_2O \otimes H_2O.$$

A linear logician might write:

$$O_2 \otimes (H_2 \otimes H_2) \vdash H_2O \otimes H_2O$$

to indicate the existence of such a morphism. But, we cannot duplicate or delete molecules, so for example

$$H_2 \not\vdash H_2 \otimes H_2$$

and

$$H_2 \not\vdash I$$

where  $I$  is the unit for the tensor product: not iodine, but “no molecules at all”.

In short, ordinary chemical reactions are morphisms in a symmetric monoidal category where objects are collections of molecules. As chemists normally conceive of it, this category is not closed. So, it obeys an even more limited system of logic than the one we have been discussing, a system lacking the connective  $\multimap$ . To get a closed category—in fact a compact one—we need to remember one of the great discoveries of twentieth-century physics: *antimatter*. This lets us define  $Y \multimap Z$  to be “anti- $Y$  and  $Z$ ”:

$$Y \multimap Z = Y^* \otimes Z.$$

Then the currying rule holds:

$$\frac{Y \otimes X \vdash Z}{X \vdash Y^* \otimes Z}$$

Most chemists don’t think about antimatter very often—but particle physicists do. They don’t use the notation of linear logic or category theory, but they know perfectly well that since a neutrino and a neutron can collide and turn into a proton and an electron:

$$\nu \otimes n \vdash p \otimes e,$$

then a neutron can turn into a antineutrino together with a proton and an electron:

$$n \vdash \nu^* \otimes (p \otimes e).$$

This is an instance of the currying rule, rule (c).

### 2.3.3 Logical Theories from Categories

We have sketched how different systems of logic naturally arise from different types of categories. To illustrate this idea, we introduced a system of logic with inference rules coming from ways to get new morphisms from old in a *closed symmetric monoidal category*. One could substitute many other types of categories here, and get other systems of logic.

To tighten the connection between proof theory and category theory, we shall now describe a recipe to get a logical theory from any closed symmetric monoidal category. For this, we shall now use  $X \vdash Y$  to denote the *set* of proofs—or actually, equivalence classes of proofs—leading from the assumption  $X$  to the conclusion  $Y$ . This is a change of viewpoint. Previously we would write  $X \vdash Y$  when this set of proofs was nonempty; otherwise we would write  $X \not\vdash Y$ . The advantage of treating  $X \vdash Y$  as a set is that this set is precisely what a category theorist would call  $\text{hom}(X, Y)$ : a homset in a category.

If we let  $X \vdash Y$  stand for a homset, an inference rule becomes a function from a product of homsets to a single homset. For example, the cut rule

$$\frac{X \vdash Y \quad Y \vdash Z}{X \vdash Z} \text{ (}\circ\text{)}$$

becomes another way of talking about the composition function

$$\circ_{X,Y,Z}: \text{hom}(X, Y) \times \text{hom}(Y, Z) \rightarrow \text{hom}(X, Z),$$

while the identity rule

$$\frac{}{X \vdash X} \text{ (i)}$$

becomes another way of talking about the function

$$i_X: 1 \rightarrow \text{hom}(X, X)$$

that sends the single element of the set 1 to the identity morphism of  $X$ . (Note: the set 1 is a *zero-fold* product of homsets.)

Next, if we let inference rules be certain functions from products of homsets to homsets, deductions become more complicated functions of the same sort built from these basic ones. For example, this deduction:

$$\frac{\frac{\frac{}{X \otimes I \vdash X \otimes I} \text{ (i)}}{X \otimes I \vdash X} \text{ (r)}}{(X \otimes I) \otimes Y \vdash X \otimes Y} \text{ (}\otimes\text{)} \quad \frac{}{Y \vdash Y} \text{ (i)}$$

specifies a function from  $1$  to  $\text{hom}((X \otimes I) \otimes Y, X \otimes Y)$ , built from the basic functions indicated by the labels at each step. This deduction:

$$\frac{\frac{\frac{}{(X \otimes I) \otimes Y \vdash (X \otimes I) \otimes Y} \text{(i)}}{(X \otimes I) \otimes Y \vdash X \otimes (I \otimes Y)} \text{(a)}}{(X \otimes I) \otimes Y \vdash X \otimes Y} \text{(o)}}{\frac{\frac{\frac{}{I \otimes Y \vdash I \otimes Y} \text{(i)}}{I \otimes Y \vdash Y} \text{(r)}}{X \otimes (I \otimes Y) \vdash X \otimes Y} \text{(o)}}{X \vdash X} \text{(i)}}{(X \otimes I) \otimes Y \vdash X \otimes Y} \text{(o)}$$

gives another function from  $1$  to  $\text{hom}((X \otimes I) \otimes Y, X \otimes Y)$ .

If we think of deductions as giving functions this way, the question arises when two such functions are equal. In the example just mentioned, the triangle equation in the definition of monoidal category (Definition 7):

$$\begin{array}{ccc} (X \otimes I) \otimes Y & \xrightarrow{a_{X,I,Y}} & X \otimes (I \otimes Y) \\ & \searrow r_X \otimes 1_Y & \swarrow 1_X \otimes l_Y \\ & X \otimes Y & \end{array}$$

says these two functions *are* equal. Indeed, the triangle equation is precisely the statement that these two functions agree! (We leave this as an exercise for the reader.)

So: even though two deductions may look quite different, they may give the same function from a product of homsets to a homset if we demand that these are homsets in a closed symmetric monoidal category. This is why we think of  $X \multimap Y$  as a set of *equivalence classes* of proofs, rather than proofs: it is forced on us by our desire to use category theory. We could get around this by using a 2-category with proofs as morphisms and “equivalences between proofs” as 2-morphisms [93, 94]. This would lead us further to the right in the Periodic Table (Table 2.3). But let us restrain ourselves and make some definitions formalizing what we have done so far.

From now on we shall call the objects  $X, Y, \dots$  “propositions”, even though we have seen they may represent more general resources. Also, purely for the sake of brevity, we use the term “proof” to mean “equivalence class of proofs”. The equivalence relation must be coarse enough to make the equations in the following definitions hold:

**Definition 19** A closed monoidal theory consists of the following:

- A collection of **propositions**. The collection must contain a proposition  $I$ , and if  $X$  and  $Y$  are propositions, then so are  $X \otimes Y$  and  $X \multimap Y$ .
- For every pair of propositions  $X, Y$ , a set  $X \vdash Y$  of **proofs** leading from  $X$  to  $Y$ . If  $f \in X \vdash Y$ , then we write  $f: X \rightarrow Y$ .
- Certain functions, written as **inference rules**:

$$\begin{array}{c}
\frac{}{X \vdash X} \text{ (i)} \qquad \frac{X \vdash Y \quad Y \vdash Z}{X \vdash Z} \text{ (o)} \\
\frac{W \vdash X \quad Y \vdash Z}{W \otimes Y \vdash X \otimes Z} \text{ (\otimes)} \qquad \frac{W \vdash (X \otimes Y) \otimes Z}{W \vdash X \otimes (Y \otimes Z)} \text{ (a)} \\
\frac{X \vdash I \otimes Y}{X \vdash Y} \text{ (l)} \qquad \frac{X \vdash Y \otimes I}{X \vdash Y} \text{ (r)} \\
\frac{X \otimes Y \vdash Z}{Y \vdash X \multimap Z} \text{ (c)}
\end{array}$$

A double line means that the function is invertible. So, for example, for each triple  $X, Y, Z$  we have a function

$$\circ_{X,Y,Z}: (X \vdash Y) \times (Y \vdash Z) \rightarrow (X \vdash Z)$$

and a bijection

$$c_{X,Y,Z}: (X \otimes Y \vdash Z) \rightarrow (Y \vdash X \multimap Z).$$

- Certain equations that must be obeyed by the inference rules. The inference rules (o) and (i) must obey equations describing associativity and the left and right unit laws. Rule (⊗) must obey an equation saying it is a functor. Rules (a), (l), (r), and (c) must obey equations saying they are natural transformations. Rules (a), (l), (r) and (⊗) must also obey the triangle and pentagon equations.

**Definition 20** A **closed braided monoidal theory** is a closed monoidal theory with this additional inference rule:

$$\frac{W \vdash X \otimes Y}{W \vdash Y \otimes X} \text{ (b)}$$

We demand that this rule give a natural transformation satisfying the hexagon equations.

**Definition 21** A **closed symmetric monoidal theory** is a closed braided monoidal theory where the rule (b) is its own inverse.

These are just the usual definitions of various kinds of closed category—monoidal, braided monoidal and symmetric monoidal—written in a new style. This new style lets us *build such categories from logical systems*. To do this, we take the objects to be propositions and the morphisms to be equivalence classes of proofs, where the equivalence relation is generated by the equations listed in the definitions above.

However, the full advantages of this style only appear when we dig deeper into proof theory, and generalize the expressions we have been considering:

$$X \vdash Y$$

to “sequents” like this:

$$X_1, \dots, X_n \vdash Y.$$

Loosely, we can think of such a sequent as meaning

$$X_1 \otimes \dots \otimes X_n \vdash Y.$$

The advantage of sequents is that they let us use inference rules that—except for the cut rule and the identity rule—have the “subformula property” mentioned near the end of Sect. 2.3.1.

Formulated in terms of these inference rules, the logic of closed symmetric monoidal categories goes by the name of “multiplicative intuitionistic linear logic”, or MILL for short [52, 91]. There is a “cut elimination” theorem for MILL, which says that with a suitable choice of other inference rules, the cut rule becomes redundant: any proof that can be done with it can be done without it. This is remarkable, since the cut rule corresponds to *composition of morphisms* in a category. One consequence is that in the free symmetric monoidal closed category on any set of objects, the set of morphisms between any two objects is *finite*. There is also a decision procedure to tell when two morphisms are equal. For details, see Trimble’s thesis [105] and the papers by Jay [58] and Soloviev [100]. Also see Kelly and Mac Lane’s coherence theorem for closed symmetric monoidal categories [67], and the related theorem for compact symmetric monoidal categories [68].

MILL is just one of many closely related systems of logic. Most include extra features, but some *subtract* features. Here are just a few examples:

- Algebraic theories. In his famous thesis, Lawvere [75] defined an **algebraic theory** to be a cartesian category where every object is an  $n$ -fold cartesian power  $X \times \dots \times X$  ( $n \geq 0$ ) of a specific object  $X$ . He showed how such categories regarded as logical theories of a simple sort—the sort that had previously been studied in “universal algebra” [26]. This work initiated the categorical approach to logic which we have been sketching here. Crole’s book [35] gives a gentle introduction to algebraic theories as well as some richer logical systems. More generally, we can think of any cartesian category as a generalized algebraic theory.
- Intuitionistic linear logic (ILL). ILL supplements MILL with the operations familiar from intuitionistic logic, as well as an operation  $!$  turning any proposition (or resource)  $X$  into an “indefinite stock of copies of  $X$ ”. Again there is a nice category-theoretic interpretation. Bierman’s thesis [24] gives a good overview, including a proof of cut elimination for ILL and a proof of the result, originally due to Girard, that intuitionistic logic can be embedded in ILL.
- Linear logic (LL). For full-fledged linear logic, the online review article by Di Cosmo and Miller [39] is a good place to start. For more, try the original paper by Girard [47] and the book by Troelstra [106]. Blute and Scott’s review article [25] serves as a Rosetta Stone for linear logic and category theory, and so do the lectures notes by Schalk [91].

- Intuitionistic Logic (IL). Lambek and Scott’s classic book [73] is still an excellent introduction to intuitionistic logic and cartesian closed categories. The online review article by Moschovakis [83] contains many suggestions for further reading.

To conclude, let us say precisely what an “inference rule” amounts to in the setup we have described. We have said it gives a function from a product of homsets to a homset. While true, this is not the last word on the subject. After all, instead of treating the propositions appearing in an inference rule as *fixed*, we can treat them as *variable*. Then an inference rule is really a “schema” for getting new proofs from old. How do we formalize this idea?

First we must realize that  $X \vdash Y$  is not just a set: it is a set *depending in a functorial way* on  $X$  and  $Y$ . As noted in Definition 14, there is a functor, the “hom functor”

$$\text{hom}: C^{\text{op}} \times C \rightarrow \text{Set},$$

sending  $(X, Y)$  to the homset  $\text{hom}(X, Y) = X \vdash Y$ . To look like logicians, let us write this functor as  $\vdash$ .

Viewed in this light, most of our inference rules are *natural transformations*. For example, rule (a) is a natural transformation between two functors from  $C^{\text{op}} \times C^3$  to  $\text{Set}$ , namely the functors

$$(W, X, Y, Z) \mapsto W \vdash (X \otimes Y) \otimes Z$$

and

$$(W, X, Y, Z) \mapsto W \vdash X \otimes (Y \otimes Z).$$

This natural transformation turns any proof

$$f: W \rightarrow (X \otimes Y) \otimes Z$$

into the proof

$$a_{X,Y,Z} f: W \rightarrow X \otimes (Y \otimes Z).$$

The fact that this transformation is *natural* means that it changes in a systematic way as we vary  $W, X, Y$  and  $Z$ . The commuting square in the definition of natural transformation, Definition 4, makes this precise.

Rules (l), (r), (b) and (c) give natural transformations in a very similar way. The  $(\otimes)$  rule gives a natural transformation between two functors from  $C^{\text{op}} \times C \times C^{\text{op}} \times C$  to  $\text{Set}$ , namely

$$(W, X, Y, Z) \mapsto (W \vdash X) \times (Y \vdash Z)$$

and

$$(W, X, Y, Z) \mapsto W \otimes Y \vdash X \otimes Z.$$

This natural transformation sends any element  $(f, g) \in \text{hom}(W, X) \times \text{hom}(Y, Z)$  to  $f \otimes g$ .

The identity and cut rules are different: they *do not* give natural transformations, because the top line of these rules has a different number of variables than the bottom line! Rule (i) says that for each  $X \in C$  there is a function

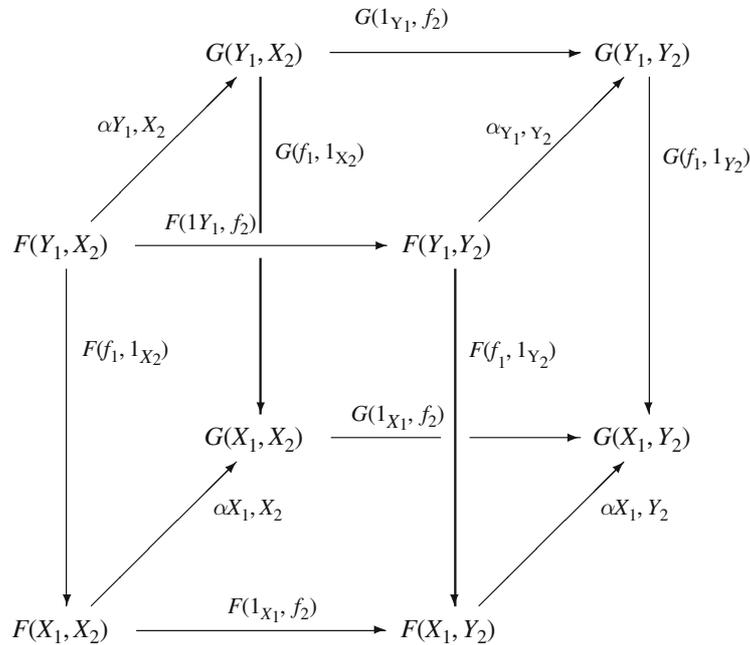
$$i_X: 1 \rightarrow X \vdash X$$

picking out the identity morphism  $1_X$ . What would it mean for this to be natural in  $X$ ? Rule (o) says that for each triple  $X, Y, Z \in C$  there is a function

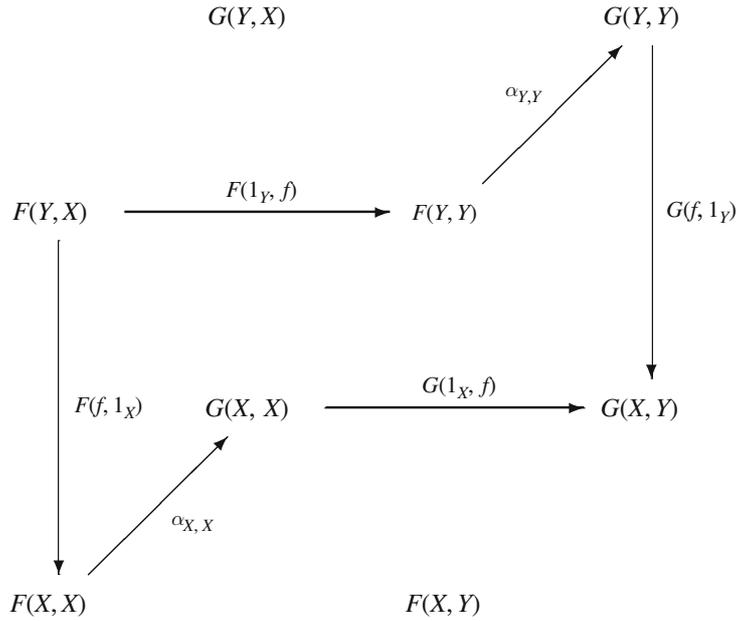
$$\circ: (X \vdash Y) \times (Y \vdash Z) \rightarrow X \vdash Z.$$

What would it mean for this to be natural in  $X, Y$  and  $Z$ ? The answer to both questions involves a generalization of natural transformations called “dinatural” transformations [77].

As noted in Definition 4, a natural transformation  $\alpha: F \Rightarrow G$  between two functors  $F, G: C \rightarrow D$  makes certain squares in  $D$  commute. If in fact  $C = C_1^{\text{op}} \times C_2$ , then we actually obtain commuting cubes in  $D$ . Namely, the natural transformation  $\alpha$  assigns to each object  $(X_1, X_2)$  a morphism  $\alpha_{X_1, X_2}$  such that for any morphism  $(f_1: Y_1 \rightarrow X_1, f_2: X_2 \rightarrow Y_2)$  in  $C$ , this cube commutes:



If  $C_1 = C_2$ , we can choose a single object  $X$  and a single morphism  $f: X \rightarrow Y$  and use it in both slots. As shown in Fig. 2.1, there are then two paths from one corner of the cube to the antipodal corner that only involve  $\alpha$  for repeated arguments: that is,  $\alpha_{X, X}$  and  $\alpha_{Y, Y}$ , but not  $\alpha_{X, Y}$  or  $\alpha_{Y, X}$ . These paths give a commuting hexagon.

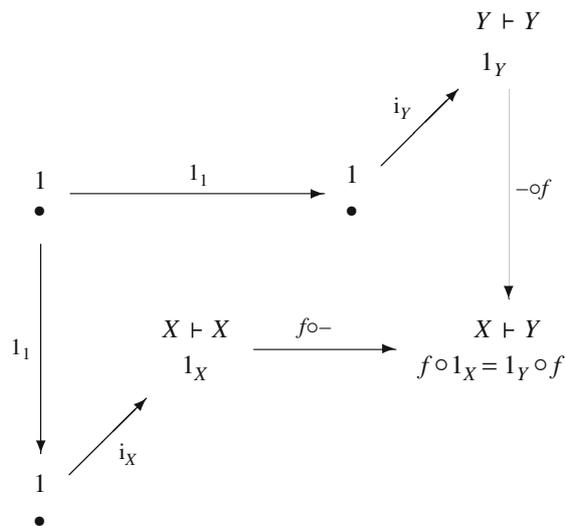


**Fig. 2.1** A natural transformation between functors  $F, G: C^{\text{op}} \times C \rightarrow D$  gives a commuting cube in  $D$  for any morphism  $f: X \rightarrow Y$ , and there are two paths around the cube that only involve  $\alpha$  for repeated arguments

This motivates the following:

**Definition 22** A **dinatural transformation**  $\alpha: F \rightrightarrows G$  between functors  $F, G: C^{\text{op}} \times C \rightarrow D$  assigns to every object  $X$  in  $C$  a morphism  $\alpha_X: F(X, X) \rightarrow G(X, X)$  in  $D$  such that for every morphism  $f: X \rightarrow Y$  in  $C$ , the hexagon in Fig. 2.1 commutes.

In the case of the identity rule, this commuting hexagon follows from the fact that the identity morphism is a left and right unit for composition: see Fig. 2.2. For the cut rule, this commuting hexagon says that composition is associative: see Fig. 2.3.



**Fig. 2.2** Dinaturality of the (i) rule, where  $f: X \rightarrow Y$ . Here  $\bullet \in 1$  denotes the one element of the one-element set

$$\begin{array}{ccccc}
 & & & & X \vdash Z \\
 & & & & h \circ (f \circ g) \\
 & & & \nearrow \circ & \downarrow 1_{X \vdash Z} \\
 (X \vdash W) \times (Y \vdash Z) & \xrightarrow{(f \circ -, 1_{Y \vdash Z})} & (X \vdash Y) \times (Y \vdash Z) & & \\
 (g, h) & & (f \circ g, h) & & \\
 \downarrow (1_{X \vdash W}, - \circ f) & & & & \\
 (X \vdash W) \times (W \vdash Z) & \nearrow \circ & X \vdash Z & \xrightarrow{1_{X \vdash Z}} & X \vdash Z \\
 (g, h \circ f) & & (h \circ f) \circ g & & (h \circ f) \circ g = h \circ (f \circ g)
 \end{array}$$

**Fig. 2.3** Dinaturality of the cut rule, where  $f: W \rightarrow Y$ ,  $g: X \rightarrow W$ ,  $h: Y \rightarrow Z$

So, in general, the sort of logical theory we are discussing involves:

- A *category*  $C$  of propositions and proofs.
- A *functor*  $\vdash: C^{\text{op}} \times C \rightarrow \text{Set}$  sending any pair of propositions to the set of proofs leading from one to the other.
- A set of *dinatural transformations* describing inference rules.

## 2.4 Computation

### 2.4.1 Background

In the 1930s, while Turing was developing what are now called “Turing machines” as a model for computation, Church and his student Kleene were developing a different model, called the “lambda calculus” [30, 69]. While a Turing machine can be seen as an idealized, simplified model of computer *hardware*, the lambda calculus is more like a simple model of *software*.

By now there are many careful treatments of the lambda calculus in the literature, from Barendregt’s magisterial tome [18] to the classic category-theoretic treatment of Lambek and Scott [73], to Hindley and Seldin’s user-friendly introduction [55] and Selinger’s elegant free online notes [96]. So, we shall content ourselves with a quick sketch.

Poetically speaking, the lambda calculus describes a universe where everything is a program and everything is data: *programs are data*. More prosaically, everything is a “ $\lambda$ -term”, or “term” for short. These are defined inductively:

- **Variables:** there is a countable set of “variables”  $x, y, z, \dots$  which are all terms.
- **Application:** if  $f$  and  $t$  are terms, we can “apply”  $f$  to  $t$  and obtain a term  $f(t)$ .
- **Lambda-abstraction:** if  $x$  is a variable and  $t$  is a term, there is a term  $(\lambda x.t)$ .

Let us explain the meaning of application and lambda-abstraction. Application is simple. Since “programs are data”, we can think of any term either as a program or a piece of data. Since we can apply programs to data and get new data, we can apply any term  $f$  to any other term  $t$  and get a new term  $f(t)$ .

Lambda-abstraction is more interesting. We think of  $(\lambda x.t)$  as the program that, given  $x$  as input, returns  $t$  as output. For example, consider

$$(\lambda x.x(x)).$$

This program takes any program  $x$  as input and returns  $x(x)$  as output. In other words, it applies any program to itself. So, we have

$$(\lambda x.x(x))(s) = s(s)$$

for any term  $s$ .

More generally, if we apply  $(\lambda x.t)$  to any term  $s$ , we should get back  $t$ , but with  $s$  substituted for each free occurrence of the variable  $x$ . This fact is codified in a rule called **beta reduction**:

$$(\lambda x.t)(s) = t[s/x]$$

where  $t[s/x]$  is the term we get by taking  $t$  and substituting  $s$  for each free occurrence of  $x$ . But beware: this rule is not an equation in the usual mathematical sense. Instead, it is a “rewrite rule”: given the term on the left, we are allowed to rewrite it and get the term on the right. Starting with a term and repeatedly applying rewrite rules is how we take a program and let it run!

There are two other rewrite rules in the lambda calculus. If  $x$  is a variable and  $t$  is a term, the term

$$(\lambda x.t(x))$$

stands for the program that, given  $x$  as input, returns  $t(x)$  as output. But this is just a fancy way of talking about the program  $t$ . So, the lambda calculus has a rewrite rule called **eta reduction**, saying

$$(\lambda x.t(x)) = t.$$

The third rewrite rule is **alpha conversion**. This allows us to replace a bound variable in a term by another variable. For example:

$$(\lambda x.x(x)) = (\lambda y.y(y))$$

since  $x$  is “bound” in the left-hand expression by its appearance in “ $\lambda x$ ”. In other words,  $x$  is just a dummy variable; its name is irrelevant, so we can replace it with  $y$ . On the other hand,

$$(\lambda x.y(x)) \neq (\lambda x.z(x)).$$

We cannot replace the variable  $y$  by the variable  $z$  here, since this variable is “free”, not bound. Some care must be taken to make the notions of free and bound variables precise, but we shall gloss over this issue, referring the reader to the references above for details.

The lambda calculus is a very simple formalism. Amazingly, starting from just this, Church and Kleene were able to build up Boolean logic, the natural numbers, the usual operations of arithmetic, and so on. For example, they defined “Church numerals” as follows:

$$\begin{aligned}\bar{0} &= (\lambda f.(\lambda x.x)) \\ \bar{1} &= (\lambda f.(\lambda x.f(x))) \\ \bar{2} &= (\lambda f.(\lambda x.f(f(x)))) \\ \bar{3} &= (\lambda f.(\lambda x.f(f(f(x))))))\end{aligned}$$

and so on. Note that  $f$  is a variable above. Thus, the Church numeral  $\bar{n}$  is the program that “takes any program to the  $n$ th power”: if you give it any program  $f$  as input, it returns the program that applies  $f$   $n$  times to whatever input  $x$  it receives.

To get a feeling for how we can define arithmetic operations on Church numerals, consider

$$\lambda g.\bar{3}(\bar{2}(g)).$$

This program takes any program  $g$ , squares it, and then cubes the result. So, it raises  $g$  to the sixth power. This suggests that

$$\lambda g.\bar{3}(\bar{2}(g)) = \bar{6}.$$

Indeed this is true. If we treat the definitions of Church numerals as reversible rewrite rules, then we can start with the left side of the above equation and grind away using rewrite rules until we reach the right side:

$$\begin{aligned}(\lambda g.\bar{3}(\bar{2}(g))) &= (\lambda g.\bar{3}((\lambda f.(\lambda x.f(f(x)))))(g)) && \text{def. of } \bar{2} \\ &= (\lambda g.\bar{3}(\lambda x.g(g(x)))) && \text{beta} \\ &= (\lambda g.(\lambda f.(\lambda x.f(f(f(x)))))(\lambda x.g(g(x)))) && \text{def. of } \bar{3} \\ &= (\lambda g.(\lambda x.(\lambda x.g(g(x))))((\lambda x.g(g(x)))(\lambda x.g(g(x))(x)))) && \text{beta} \\ &= (\lambda g.(\lambda x.(\lambda x.g(g(x))))((\lambda g.g(g(x)))(g(g(x)))))) && \text{beta} \\ &= (\lambda g.(\lambda x.(\lambda x.g(g(x))))(g(g(g(x)))))) && \text{beta} \\ &= (\lambda g.(\lambda x.g(g(g(g(x)))))) && \text{beta} \\ &= \bar{6} && \text{def. of } \bar{6}\end{aligned}$$

If this calculation seems mind-numbing, that is precisely the point: it resembles the inner workings of a computer. We see here how the lambda calculus can serve as a programming language, with each step of computation corresponding to a rewrite rule.

Of course, we got the answer  $\bar{6}$  because  $3 \times 2 = 6$ . Generalizing from this example, we can define a program called “times” that multiplies Church numerals:

$$\text{times} = (\lambda a.(\lambda b.(\lambda x.a(b(x)))))$$

For example,

$$\text{times}(\bar{3})(\bar{2}) = \bar{6}.$$

The enterprising reader can dream up similar programs for the other basic operations of arithmetic. With more cleverness, Church and Kleene were able to write terms corresponding to more complicated functions. They eventually came to believe that *all* computable functions  $f: \mathbb{N} \rightarrow \mathbb{N}$  can be defined in the lambda calculus.

Meanwhile, Gödel was developing another approach to computability, the theory of “recursive functions”. Around 1936, Kleene proved that the functions definable in the lambda calculus were the same as Gödel’s recursive functions. In 1937 Turing described his “Turing machines”, and used these to give yet another definition of computable functions. This definition was later shown to agree with the other two. Thanks to this and other evidence, it is now widely accepted that the lambda calculus can define *any* function that can be computed by *any* systematic method. We say it is “Turing complete”.

After this burst of theoretical work, it took a few decades for programmable computers to actually be built. It took even longer for computer scientists to profit from Church and Kleene’s insights. This began around 1958, when McCarthy invented the programming language Lisp, based on the lambda calculus [80]. In 1965, an influential paper by Landin [74] pointed out a powerful analogy between the lambda calculus and the language ALGOL. These developments led to a renewed interest in the lambda calculus which continues to this day. By now, a number of computer languages are explicitly based on ideas from the lambda calculus. The most famous of these include Lisp, ML and Haskell. These languages, called “functional programming languages”, are beloved by theoretical computer scientists for their conceptual clarity. In fact, for many years, everyone majoring in computer science at MIT has been required to take an introductory course that involves programming in Scheme, a dialect of Lisp. The cover of the textbook for this course [1] even has a big  $\lambda$  on the cover!

We should admit that languages of a different sort—“imperative programming languages”—are more popular among working programmers. Examples include FORTRAN, BASIC, and C. In imperative programming, a program is a series of instructions that tell the computer what to do. By contrast, in functional programming, a program simply describes a function. To run the program, we apply it to an input. So, as in the lambda calculus, “application” is a fundamental operation in functional programming. If we combine application with lambda abstraction, we obtain a language powerful enough to compute any computable function.

However, most functional programming languages are more regimented than the original lambda calculus. As we have seen, in the lambda calculus as originally developed by Church and Kleene, any term can be applied to any other. In real life,

programming involves many kinds of data. For example, suppose we are writing a program that involves days of the week. It would not make sense to write

$$\text{times}(\bar{3})(\text{Tuesday})$$

because Tuesday is not a number. We might choose to represent Tuesday by a number in some program, but doubling that number doesn't have a good interpretation: is the first day of the week Sunday or Monday? Is the week indexed from zero or one? These are arbitrary choices that affect the result. We could let the programmer make the choices, but the resulting unstructured framework easily leads to mistakes.

It is better to treat data as coming in various "types", such as integers, floating-point numbers, alphanumeric strings, and so on. Thus, whenever we introduce a variable in a program, we should make a "type declaration" saying what type it is. For example, we might write:

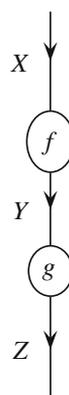
$$\text{Tuesday} : \text{day}$$

This notation is used in Ada, Pascal and some other languages. Other notations are also in widespread use. Then, our system should have a "type checker" (usually part of the compiler) that complains if we try to apply a program to a piece of data of the wrong type.

Mathematically, this idea is formalized by a more sophisticated version of the lambda calculus: the "typed" lambda calculus, where every term has a type. This idea is also fundamental to category theory, where every morphism is like a black box with input and output wires of specified types:



and it makes no sense to hook two black boxes together unless the output of the first has the same type as the input of the next:



Indeed, there is a deep relation between the typed lambda calculus and cartesian closed categories. This was discovered by Lambek in 1980 [72]. Quite roughly speaking, a “typed lambda-theory” is a very simple functional programming language with a specified collection of basic data types from which other more complicated types can be built, and a specified collection of basic terms from which more complicated terms can be built. The data types of this language are *objects* in a cartesian closed category, while the programs—that is, terms—give *morphisms*!

Here we are being a bit sloppy. Recall from Sect. 2.3.3 that in logic we can build closed monoidal categories where the morphisms are equivalence classes of proofs. We need to take equivalence classes for the axioms of a closed monoidal category to hold. Similarly, to get closed monoidal categories from computer science, we need the morphisms to be equivalence classes of terms. Two terms count as equivalent if they differ by rewrite rules such as beta reduction, eta reduction and alpha conversion. As we have seen, these rewrites represent the steps whereby a program carries out its computation. For example, in the original “untyped” lambda calculus, the terms  $\text{times}(\bar{3})(\bar{2})$  and  $\bar{6}$  differ by rewrite rules, but they give the same morphism. So, when we construct a cartesian closed category from a typed lambda-theory, we *neglect the actual process of computation*. To remedy this we should work with a cartesian closed 2-category which has:

- types as objects,
- terms as morphisms,
- equivalence classes of rewrites as 2-morphisms.

For details, see the work of Seely [93, 94], Hilken [54], and Melliés [79]. Someday this work will be part of the larger  $n$ -categorical Rosetta Stone mentioned at the end of Sect. 2.2.5.

In any event, Lambek showed that every typed lambda-theory gives a cartesian closed category—and conversely, every cartesian closed category gives a typed lambda-theory. This discovery led to a rich line of research blending category theory and computer science. There is no way we can summarize the resulting enormous body of work, though it constitutes a crucial aspect of the Rosetta Stone. Two good starting points for further reading are the textbook by Crole [35] and the online review article by Scott [89].

In what follows, our goal is more limited. First, in Sect. 2.4.2, we explain how every “typed lambda-theory” gives a cartesian closed category, and conversely. We follow the treatment of Lambek and Scott [73], in a somewhat simplified form. Then, in Sect. 2.4.3, we describe how every “linear type theory” gives a closed symmetric monoidal category, and conversely.

The idea here is roughly that a “linear type theory” is a programming language suitable for *both classical and quantum computation*. This language differs from the typed lambda calculus in that it forbids duplication and deletion of data except when expressly permitted. The reason is that while every object in a cartesian category comes equipped with “duplication” and “deletion” morphisms:

$$\Delta_X : X \rightarrow X \otimes X, \quad !_X : X \rightarrow 1,$$

a symmetric monoidal category typically lacks these. As we saw in Sect. 2.2.3, a great example is the category  $\text{Hilb}$  with its usual tensor product. So, a programming language suitable for quantum computation should not assume we can duplicate all types of data [29, 110].

Various versions of “quantum” or “linear” lambda calculus have already been studied, for example by Benton, Bierman de Paiva and Hyland [22], Dorca and van Tonder [108], and Selinger and Valiron [98]. Abramsky and Tzevelekos sketch a version in their paper in this volume [6]. We instead explain the ‘linear type theories’ developed by Simon Ambler in his 1991 thesis [7].

### 2.4.2 The Typed Lambda Calculus

Like the original “untyped” lambda calculus explained above, the typed lambda calculus uses terms to represent both programs and data. However, now every term has a specific type. A program that inputs data of type  $X$  and outputs data of type  $Y$  is said to be of type  $X \multimap Y$ . So, we can only apply a term  $s$  to a term  $t$  of type  $X$  if  $s$  is of type  $X \multimap Y$  for some  $Y$ . In this case  $s(t)$  is a well-defined term of type  $Y$ . We call  $X \multimap Y$  a **function type**.

Whenever we introduce a variable, we must declare its type. We write  $t : X$  to mean that  $t$  is a term of type  $X$ . So, in lambda abstraction, we no longer simply write expressions like  $(\lambda x . t)$ . Instead, if  $x$  is a variable of type  $X$ , we write

$$(\lambda x : X . t).$$

For example, here is a simple program that takes a program of type  $X \multimap X$  and “squares” it:

$$(\lambda f : X \multimap X . (\lambda x : X . f(f(x)))).$$

In the original lambda calculus, all programs take a single piece of data as input. In other words, they compute unary functions. This is no real limitation, since we can handle functions that take more than one argument using a trick called “currying”, discussed in Sect. 2.2.6 This turns a function of several arguments into a function that takes the first argument and returns a function of the remaining arguments. We saw an example in the last section: the program “times”. For example,  $\text{times}(\bar{3})$  is a program that multiplies by 3, so  $\text{times}(\bar{3})(\bar{2}) = \bar{6}$ .

While making all programs compute unary functions is economical, it is not very kind to the programmer. So, in the typed lambda calculus we also introduce products: given types  $X$  and  $Y$ , there is a type  $X \times Y$  called a **product type**. We can think of a datum of type  $X \times Y$  as a pair consisting of a datum of type  $X$  and a datum of type  $Y$ . To make this intuition explicit, we insist that given terms  $s : X$  and  $t : Y$  there is a term  $(s, t) : X \times Y$ . We also insist that given a term  $u : X \times Y$

there are terms  $p(u) : X$  and  $p'(u) : Y$ , which we think of as the first and second components of the pair  $t$ . We also include rewrite rules saying:

$$\begin{aligned} (p(u), p'(u)) &= u \text{ for all } u : X \times Y, \\ p(s, t) &= s \text{ for all } s : X \text{ and } t : Y, \\ p'(s, t) &= t \text{ for all } s : X \text{ and } t : Y. \end{aligned}$$

Product types allow us to write programs that take more than one input. Even more importantly, they let us deal with programs that produce more than one output. For example, we might have a type called “integer”. Then we might want a program that takes an integer and duplicates it:

$$\text{duplicate} : \text{integer} \multimap (\text{integer} \times \text{integer})$$

Such a program is easy to write:

$$\text{duplicate} = (\lambda x : \text{integer} . (x, x)).$$

Of course this a program we should *not* be allowed to write when duplicating information is forbidden, but in this section our considerations are all “classical”, i.e., suited to cartesian closed categories.

The typed lambda calculus also has a special type called the “unit type”, which we denote as  $1$ . There is a single term of this type, which we denote as  $()$ . From the viewpoint of category theory, the need for this type is clear: a category with finite products must have not only binary products but also a terminal object (see Definition 10). For example, in the category  $\text{Set}$ , the terminal object can be taken as any one-element set, and  $()$  is the unique element of this set. It may be less clear why this type is useful in programming. One reason is that it lets us think of a constant of type  $X$  as a function of type  $1 \multimap X$ —that is, a “nullary” function, one that takes no arguments. There are some other reasons, but they go beyond the scope of this discussion. Suffice it to say that Haskell, Lisp and even widely used imperative languages such as C, C++ and Java include the unit type.

Having introduced the main ingredients of the typed lambda calculus, let us give a more formal treatment. As we shall see, a “typed lambda-theory” consists of types, terms and rewrite rules. From a typed lambda-theory we can get a cartesian closed category. The types will give objects, the terms will give morphisms, and the rewrite rules will give equations between morphisms.

First, the **types** are given inductively as follows:

- **Basic types:** There is an arbitrarily chosen set of types called **basic types**.
- **Product types:** Given types  $X$  and  $Y$ , there is a type  $X \times Y$ .
- **Function types:** Given types  $X$  and  $Y$ , there is a type  $X \multimap Y$ .
- **Unit type:** There is a type  $1$ .

There may be unexpected equations between types: for example we may have a type  $X$  satisfying  $X \times X = X$ . However, we demand that:

- If  $X = X'$  and  $Y = Y'$  then  $X \times Y = X' \times Y'$ .
- If  $X = X'$  and  $Y = Y'$  then  $X \multimap Y = X' \multimap Y'$ .

Next we define **terms**. Each term has a specific type, and if  $t$  is a term of type  $X$  we write  $t : X$ . The rules for building terms are as follows:

- **Basic terms:** For each type  $X$  there is a set of **basic terms** of type  $X$ .
- **Variables:** For each type  $X$  there is a countably infinite collection of terms of type  $X$  called **variables** of type  $X$ .
- **Application:** If  $f : X \multimap Y$  and  $t : X$  then there is a term  $f(t)$  of type  $Y$ .
- **Lambda abstraction:** If  $x$  is a variable of type  $X$  and  $t : Y$  then there is a term  $(\lambda x : X . t)$  of type  $X \multimap Y$ .
- **Pairing:** If  $s : X$  and  $t : Y$  then there is a term  $(s, t)$  of type  $X \times Y$ .
- **Projection:** If  $t : X \times X'$  then there is a term  $p(t)$  of type  $X$  and a term  $p'(t)$  of type  $X'$ .
- **Unit term:** There is a term  $()$  of type  $1$ .

Finally there are **rewrite rules** going between terms of the same type. Given any fixed set of variables  $\mathcal{S}$ , there will be rewrite rules between terms of the same type, all of whose free variables lie in the set  $\mathcal{S}$ . For our present purposes, we only need these rewrite rules to decide when two terms determine the same morphism in the cartesian closed category we shall build. So, what matters is not really the rewrite rules themselves, but the equivalence relation they generate. We write this equivalence relation as  $s \sim_{\mathcal{S}} t$ .

The relation  $\sim_{\mathcal{S}}$  can be any equivalence relation satisfying the following list of rules. In what follows,  $t[s/x]$  denotes the result of taking a term  $t$  and replacing every free occurrence of the variable  $x$  by the term  $s$ . Also, when we say ‘term’ without further qualification, we mean ‘term all of whose free variables lie in the set  $\mathcal{S}$ ’.

- **Type preservation:** If  $t \sim_{\mathcal{S}} t'$  then  $t$  and  $t'$  must be terms of the same type, all of whose free variables lie in the set  $\mathcal{S}$ .
- **Beta reduction:** Suppose  $x$  is a variable of type  $X$ ,  $s$  is a term of type  $X$ , and  $t$  is any term. If no free occurrence of a variable in  $s$  becomes bound in  $t[s/x]$ , then:

$$(\lambda x : X . t)(s) \sim_{\mathcal{S}} t[s/x].$$

- **Eta reduction:** Suppose the variable  $x$  does not appear in the term  $f$ . Then:

$$(\lambda x : X . f(x)) \sim_{\mathcal{S}} f.$$

- **Alpha conversion:** Suppose  $x$  and  $y$  are variables of type  $X$ , and no free occurrence of any variable in  $t$  becomes bound in  $t[x/y]$ . Then:

$$(\lambda x : X . t) \sim_{\mathcal{S}} (\lambda y : X . t[x/y]).$$

- **Application:** Suppose  $t$  and  $t'$  are terms of type  $X$  with  $t \sim_{\mathcal{S}} t'$ , and suppose that  $f: X \multimap Y$ . Then:

$$f(t) \sim_{\mathcal{S}} f(t').$$

- **Lambda abstraction:** Suppose  $t$  and  $t'$  are terms of type  $Y$ , all of whose free variables lie in the set  $\mathcal{S} \cup \{x\}$ . Suppose that  $t \sim_{\mathcal{S} \cup \{x\}} t'$ . Then:

$$(\lambda x: X. t) \sim_{\mathcal{S}} (\lambda x: X. t')$$

- **Pairing:** If  $u$  is a term of type  $X \times Y$  then:

$$(p(u), p'(u)) \sim_{\mathcal{S}} u.$$

- **Projection:** if  $s$  is a term of type  $X$  and  $t$  is a term of type  $Y$  then:

$$\begin{aligned} p(s, t) &\sim_{\mathcal{S}} s \\ p'(s, t) &\sim_{\mathcal{S}} t. \end{aligned}$$

- **Unit term:** If  $t$  is a term of type  $1$  then:

$$t \sim_{\mathcal{S}} ().$$

Now we can describe Lambek's classic result relating typed lambda-theories to cartesian closed categories. From a typed lambda-theory we get a cartesian closed category  $C$  for which:

- The objects of  $C$  are the types.
- The morphisms  $f: X \rightarrow Y$  of  $C$  are equivalence classes of pairs  $(x, t)$  consisting of a variable  $x: X$  and a term  $t: Y$  with no free variables except perhaps  $x$ . Here  $(x, t)$  is equivalent to  $(x, t')$  if and only if:

$$t \sim_{\{x\}} t'[x/x'].$$

- Given a morphism  $f: X \rightarrow Y$  coming from a pair  $(x, t)$  and a morphism  $g: Y \rightarrow Z$  coming from a pair  $(y, u)$  as above, the composite  $gf: X \rightarrow Z$  comes from the pair  $(x, u[t/y])$ .

We can also reverse this process and get a typed lambda-theory from a cartesian closed category. In fact, Lambek and Scott nicely explain how to construct a category of cartesian closed categories and a category of typed-lambda theories. They construct functors going back and forth between these categories and show these functors are inverses up to natural isomorphism. We thus say these categories are "equivalent" [73].

### 2.4.3 Linear Type Theories

In his thesis [7], Ambler described how to generalize Lambek’s classic result from cartesian closed categories to closed symmetric monoidal categories. To do this, he replaced typed lambda-theories with “linear type theories”. A linear type theory can be seen as a programming language suitable for both classical and quantum computation. As we have seen, in a noncartesian category like  $\text{Hilb}$ , we cannot freely duplicate or delete information. So linear type theories must prevent duplication or deletion of data *except when it is expressly allowed*.

To achieve this, linear type theories must not allow us to write a program like this:

$$(\lambda x : X . (x, x)).$$

Even a program that “squares” another program, like this:

$$(\lambda f : X \multimap X . (\lambda x : X . f(f(x)))),$$

is not allowed, since it “reuses” the variable  $f$ . On the other hand, a program that composes two programs is allowed!

To impose these restrictions, linear type theories treat variables very differently than the typed lambda calculus. In fact, in a linear type theory, any term will contain a given variable at most *once*. But linear type theories depart even more dramatically from the typed lambda calculus in another way. They make no use of lambda abstraction! Instead, they use “combinators”.

The idea of a combinator is very old: in fact, it predates the lambda calculus. Combinatory logic was born in a 1924 paper by Schönfinkel [92], and was rediscovered and extensively developed by Curry [36, 37] starting in 1927. In retrospect, we can see their work as a stripped-down version of the untyped lambda calculus that completely avoids the use of variables. Starting from a basic stock of terms called “combinators”, the only way to build new ones is application: we can apply any term  $f$  to any term  $t$  and get a term  $f(t)$ .

To build a Turing-complete programming language in such an impoverished setup, we need a sufficient stock of combinators. Remarkably, it suffices to use three. In fact it is possible to use just *one* cleverly chosen combinator—but this tour de force is not particularly enlightening, so we shall describe a commonly used set of three. The first, called  $I$ , acts like the identity, since it comes with the rewrite rule:

$$I(a) = a$$

for every term  $a$ . The second, called  $K$ , gives a constant function  $K(a)$  for each term  $a$ . In other words, it comes with a rewrite rule saying

$$K(a)(b) = a$$

for every term  $b$ . The third, called  $S$ , is the tricky one. It takes three terms, applies the first to the third, and applies the result to the second applied to the third:

$$S(a)(b)(c) = a(c)(b(c)).$$

Later it was seen that the combinator calculus can be embedded in the untyped lambda calculus as follows:

$$\begin{aligned} I &= (\lambda x.x) \\ K &= (\lambda x.(\lambda y.x)) \\ S &= (\lambda x.(\lambda y.(\lambda z.x(z)(y(z))))). \end{aligned}$$

The rewrite rules for these combinators then follow from rewrite rules in the lambda calculus. More surprisingly, any function computable using the lambda calculus can also be computed using just  $I$ ,  $K$  and  $S$ ! While we do not need this fact to understand linear type theories, we cannot resist sketching the proof, since it is a classic example of using combinators to avoid explicit use of lambda abstraction.

Note that all the variables in the lambda calculus formulas for  $I$ ,  $K$ , and  $S$  are bound variables. More generally, in the lambda calculus we define a **combinator** to be a term in which all variables are bound variables. Two combinators  $c$  and  $d$  are **extensionally equivalent** if they give the same result on any input: that is, for any term  $t$ , we can apply lambda calculus rewrite rules to  $c(t)$  and  $d(t)$  in a way that leads to the same term. There is a process called “abstraction elimination” that takes any combinator in the lambda calculus and produces an extensionally equivalent one built from  $I$ ,  $K$ , and  $S$ .

Abstraction elimination works by taking a term  $t = (\lambda x.u)$  with a single lambda abstraction and rewriting it into the form  $(\lambda x.f(x))$ , where  $f$  has no instances of lambda abstraction. Then we can apply eta reduction, which says  $(\lambda x.f(x)) = f$ . This lets us rewrite  $t$  as a term  $f$  that does not involve lambda abstraction. We shall use the notation  $[[u]]_x$  to mean “any term  $f$  satisfying  $f(x) = u$ ”.

There are three cases to consider; each case justifies the definition of one combinator:

1.  $t = (\lambda x.x)$ . We can rewrite this as  $t = (\lambda x.I(x))$ , so  $t = [[x]]_x = I$ .
2.  $t = (\lambda x.u)$ , where  $u$  does not depend on  $x$ . We can rewrite this as  $t = (\lambda x.K(u)(x))$ , so  $t = [[u]]_x = K(u)$ .
3.  $t = (\lambda x.u(v))$ , where  $u$  and  $v$  may depend on  $x$ . We can rewrite this as  $t = (\lambda x.(((u)]_x)([v]]_x))$  or  $t = (\lambda x.S([u]]_x)([v]]_x(x))$ , so  $t = S([u]]_x)([v]]_x)$ .

We can eliminate all use of lambda abstraction from any term by repeatedly using these three rules “from the inside out”. To see how this works, consider the lambda term  $t = (\lambda x.(\lambda y.y))$ , which takes two inputs and returns the second. Using the rules above we have:

$$\begin{aligned}
(\lambda x.(\lambda y.y)) &= (\lambda x.(\lambda y.[[y]]_y(y))) \\
&= (\lambda x.(\lambda y.I(y))) \\
&= (\lambda x.I) \\
&= (\lambda x.[[I]]_x(x)) \\
&= (\lambda x.K(I)(x)) \\
&= K(I).
\end{aligned}$$

We can check that it works as desired:  $K(I)(x)(y) = I(y) = y$ .

Now let us return to our main theme: linear type theories. Of the three combinators described above, only  $I$  is suitable for use in an arbitrary closed symmetric monoidal category. The reason is that  $K$  deletes data, while  $S$  duplicates it. We can see this directly from the rewrite rules they satisfy:

$$\begin{aligned}
K(a)(b) &= a \\
S(a)(b)(c) &= a(c)(b(c)).
\end{aligned}$$

Every linear type theory has a set of “basic combinators”, which neither duplicate nor delete data. Since linear type theories generalize *typed* lambda-theories, these basic combinators are typed. Ambler writes them using notation resembling the notation for morphisms in category theory.

For example, given two types  $X$  and  $Y$  in a linear type theory, there is a **tensor product type**  $X \otimes Y$ . This is analogous to a product type in the typed lambda calculus. In particular, given a term  $s$  of type  $X$  and a term  $t$  of type  $Y$ , we can combine them to form a term of type  $X \otimes Y$ , which we now denote as  $(s \otimes t)$ . We reparenthesize iterated tensor products using the following basic combinator:

$$\text{assoc}_{X,Y,Z}: (X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z).$$

This combinator comes with the following rewrite rule:

$$\text{assoc}_{X,Y,Z}((s \otimes t) \otimes u) = (s \otimes (t \otimes u))$$

for all terms  $s : X$ ,  $t : Y$  and  $u : Z$ .

Of course, the basic combinator  $\text{assoc}_{X,Y,Z}$  is just a mildly disguised version of the associator, familiar from category theory. Indeed, all the basic combinators come from natural or dinatural transformations implicit in the definition of “closed symmetric monoidal category”. In addition to these, any given linear type theory also has combinators called “function symbols”. These come from the morphisms particular to a given category. For example, suppose in some category the tensor product  $X \otimes X$  is actually the cartesian product. Then the corresponding linear type theory should have a function symbol

$$\Delta_X: X \rightarrow X \otimes X$$

which lets us duplicate data of type  $X$ , together with function symbols

$$p: X \otimes X \rightarrow X, \quad p': X \otimes X \rightarrow X$$

that project onto the first and second factors. To make sure these work as desired, we can include rewrite rules:

$$\begin{aligned} \Delta(s) &= (s \otimes s) \\ p(s \otimes t) &= s \\ p'(s \otimes t) &= t. \end{aligned}$$

So, while duplication and deletion of data is not a “built-in feature” of linear type theories, we can include it when desired.

Using combinators, we could try to design a programming language suitable for closed symmetric monoidal categories that completely avoid the use of variables. Ambler follows a different path. He retains variables in his formalism, but they play a very different—and much *simpler*—role than they do in the lambda calculus. Their only role is to help decide which terms should count as equivalent. Furthermore, lambda abstraction plays no role in linear type theories, so the whole issue of free versus bound variables does not arise! In a sense, all variables are free. Moreover, every term contains any given variable at most once.

After these words of warning, we hope the reader is ready for a more formal treatment of linear type theories. A **linear type theory** has types, combinators, terms, and rewrite rules. The types will correspond to objects in a closed symmetric monoidal category, while equivalence classes of combinators will correspond to morphisms. Terms and rewrite rules are only used to define the equivalence relation.

First, the set of **types** is defined inductively as follows:

- **Basic types:** There is an arbitrarily chosen set of types called **basic types**.
- **Product types:** Given types  $X$  and  $Y$ , there is a type  $(X \otimes Y)$ .
- **Function types:** Given types  $X$  and  $Y$ , there is a type  $(X \multimap Y)$ .
- **Trivial type:** There is a type  $I$ .

There may be equations between types, but we require that:

- If  $X = X'$  and  $Y = Y'$  then  $X \otimes Y = X' \otimes Y'$ .
- If  $X = X'$  and  $Y = Y'$  then  $X \multimap Y = X' \multimap Y'$ .

Second, a linear type theory has for each pair of types  $X$  and  $Y$  a set of **combinators** of the form  $f: X \rightarrow Y$ . These are defined by the following inductive rules:

- Given types  $X$  and  $Y$  there is an arbitrarily chosen set of combinators  $f: X \rightarrow Y$  called **function symbols**.
- Given types  $X, Y$ , and  $Z$  we have the following combinators, called **basic combinators**:

- $\text{id}_X: X \rightarrow X$
- $\text{assoc}_{X,Y,Z}: (X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z)$
- $\text{unassoc}_{X,Y,Z}: X \otimes (Y \otimes Z) \rightarrow (X \otimes Y) \otimes Z$

- $\text{braid}_{X,Y} : X \otimes Y \rightarrow Y \otimes X$
- $\text{left}_X : I \otimes X \rightarrow X$
- $\text{unleft}_X : X \rightarrow I \otimes X$
- $\text{right}_X : I \otimes X \rightarrow X$
- $\text{unright}_X : X \rightarrow I \otimes X$
- $\text{eval}_{X,Y} : X \otimes (X \multimap Y) \rightarrow Y$
- If  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are combinators, then  $(g \circ f) : X \rightarrow Z$  is a combinator.
- If  $f : X \rightarrow Y$  and  $g : X' \rightarrow Y'$  are combinators, then  $(f \otimes g) : X \otimes X' \rightarrow Y \otimes Y'$  is a combinator.
- If  $f : X \otimes Y \rightarrow Z$  is a combinator, then we can **curry**  $f$  to obtain a combinator  $\tilde{f} : Y \rightarrow (X \multimap Z)$ .

It will generally cause no confusion if we leave out the subscripts on the basic combinators. For example, we may write simply “assoc” instead of  $\text{assoc}_{X,Y,Z}$ .

Third, a linear type theory has a set of **terms** of any given type. As usual, we write  $t : X$  to say that  $t$  is a term of type  $X$ . Terms are defined inductively as follows:

- For each type  $X$  there is a countably infinite collection of **variables** of type  $X$ . If  $x$  is a variable of type  $X$  then  $x : X$ .
- There is a term  $1$  with  $1 : I$ .
- If  $s : X$  and  $t : Y$ , then there is a term  $(s \otimes t)$  with  $(s \otimes t) : X \otimes Y$ , as long as no variable appears in both  $s$  and  $t$ .
- If  $f : X \rightarrow Y$  is a combinator and  $t : X$  then there is a term  $f(t)$  with  $f(t) : Y$ .

Note that any given variable may appear at most once in a term.

Fourth and finally, a linear type theory has **rewrite rules** going between terms of the same type. As in our treatment of the typed lambda calculus, we only care here about the equivalence relation  $\sim$  generated by these rewrite rules. This equivalence relation must have all the properties listed below. In what follows, we say a term is **basic** if it contains no combinators. Such a term is just an iterated tensor product of distinct variables, such as

$$(z \otimes ((x \otimes y) \otimes w)).$$

These are the properties that the equivalence relation  $\sim$  must have:

- If  $t \sim t'$  then  $t$  and  $t'$  must be terms of the same type, containing the same variables.
- The equivalence relation is **substitutive**:
  - Given terms  $s \sim s'$ , a variable  $x$  of type  $X$ , and terms  $t \sim t'$  of type  $X$  whose variables appear in neither  $s$  nor  $s'$ , then  $s[t/x] \sim s'[t'/x]$ .
  - Given a basic term  $t$  with the same type as a variable  $x$ , if none of the variables of  $t$  appear in the terms  $s$  or  $s'$ , and  $s[t/x] \sim s'[t/x]$ , then  $s \sim s'$ .
- The equivalence relation is **extensional**: if  $f : X \multimap Y$ ,  $g : X \multimap Y$  and  $\text{eval}(t \otimes f) = \text{eval}(t \otimes g)$  for all basic terms  $t : X$ , then  $f = g$ .

- We have:
  - $\text{id}(s) \sim s$
  - $(g \circ f)(s) \sim g(f(s))$
  - $(f \otimes g)(s \otimes t) \sim (f(s) \otimes g(t))$
  - $\text{assoc}((s \otimes t) \otimes u) \sim (s \otimes (t \otimes u))$
  - $\text{unassoc}(s \otimes (t \otimes u)) \sim ((s \otimes t) \otimes u)$
  - $\text{braid}(s \otimes t) \sim (t \otimes s)$
  - $\text{left}(1 \otimes s) \sim s$
  - $\text{unleft}(s) \sim (1 \otimes s)$
  - $\text{right}(1 \otimes s) \sim s$
  - $\text{unright}(s) \sim (1 \otimes s)$
  - $\text{eval}(s \otimes \tilde{f}(t)) \sim f(s \otimes t)$

Note that terms can have variables appearing anywhere within them. For example, if  $x, y, z$  are variables of types  $X, Y$  and  $Z$ , and  $f: Y \otimes Z \rightarrow W$  is a function symbol, then

$$\text{braid}(x \otimes f(y \otimes z))$$

is a term of type  $W \otimes X$ . However, every term  $t$  is equivalent to a term of the form  $\text{cp}(t)(\text{vp}(t))$ , where  $\text{cp}(t)$  is the **combinator part** of  $t$  and  $\text{vp}(t)$  is a basic term called the **variable part** of  $t$ . For example, the above term is equivalent to

$$\text{braid} \circ (\text{id} \otimes (f \circ (\text{id} \otimes \text{id}))) (x \otimes (y \otimes z)).$$

The combinator and variable parts can be computed inductively as follows:

- If  $x$  is a variable of type  $X$ ,  $\text{cp}(x) = \text{id}: X \rightarrow X$ .
- $\text{cp}(1) = \text{id}: I \rightarrow I$ .
- For any terms  $s$  and  $t$ ,  $\text{cp}(s \otimes t) = \text{cp}(s) \otimes \text{cp}(t)$ .
- For any term  $s: X$  and any combinator  $f: X \rightarrow Y$ ,  $\text{cp}(f(s)) = f \circ \text{cp}(s)$ .
- If  $x$  is a variable of type  $X$ ,  $\text{vp}(x) = x$ .
- $\text{vp}(1) = 1$ .
- For any terms  $s$  and  $t$ ,  $\text{vp}(s \otimes t) = \text{vp}(s) \otimes \text{vp}(t)$ .
- For any term  $s: X$  and any combinator  $f: X \rightarrow Y$ ,  $\text{vp}(f(s)) = \text{vp}(s)$ .

Now, suppose that we have a linear type theory. Ambler's first main result is this: there is a symmetric monoidal category where objects are types and morphisms are equivalence classes of combinators. The equivalence relation on combinators is defined as follows: two combinators  $f, g: X \rightarrow Y$  are equivalent if and only if

$$f(t) \sim g(t)$$

for some basic term  $t$  of type  $X$ . In fact, Ambler shows that  $f(t) \sim g(t)$  for *some* basic term  $t: X$  if and only if  $f(t) \sim g(t)$  for *all* such basic terms.

Ambler's second main result describes how we can build a linear type theory from any closed symmetric monoidal category, say  $C$ . Suppose  $C$  has composition  $\square$ , tensor product  $\bullet$ , internal hom  $\dashv$ , and unit object  $\iota$ . We let the basic types of

our linear type theory be the objects of  $C$ . We take as equations between types those generated by:

- $\iota = I$
- $A \bullet B = A \otimes B$
- $A \dashv B = A \multimap B$

We let the function symbols be all the morphisms of  $C$ . We take as our equivalence relation on terms the smallest allowed equivalence relation such that:

- $1_A(x) \sim A$
- $(g \square f)(x) \sim g(f(x))$
- $(f \bullet g)(x \otimes y) \sim (f(x) \otimes g(y))$
- $a_{A,B,C}((x \otimes y) \otimes z) \sim (x \otimes (y \otimes z))$
- $b_{A,B}(x \otimes y) \sim (y \otimes x)$
- $l_A(1 \otimes x) \sim x$
- $r_A(x \otimes 1) \sim x$
- $\text{ev}_{A,B}(x \otimes \tilde{f}(y)) \sim f(x \otimes y)$

Then we define

- $\text{id} = 1$
- $\text{assoc} = a$
- $\text{unassoc} = a^{-1}$
- $\text{braid} = b$
- $\text{left} = l$
- $\text{unleft} = l^{-1}$
- $\text{right} = r$
- $\text{unright} = r^{-1}$
- $\text{eval} = \text{ev}$
- $g \circ f = g \square f$

and we're done!

Ambler also shows that this procedure is the “inverse” of his procedure for turning linear type theories into closed symmetric monoidal categories. More precisely, he describes a category of closed symmetric monoidal categories (which is well-known), and also a category of linear type theories. He constructs functors going back and forth between these, based on the procedures we have sketched, and shows that these functors are inverses up to natural isomorphism. So, these categories are “equivalent”.

In this section we have focused on closed symmetric monoidal categories. What about closed categories that are just braided monoidal, or merely monoidal? While we have not checked the details, we suspect that programming languages suited to these kinds of categories can be obtained from Ambler's formalism by removing various features. To get the braided monoidal case, the obvious guess is to remove Ambler's rewrite rule for the ‘braid’ combinator and add two rewrite rules corresponding to the hexagon equations (see Sect. 2.2.4 for these). To get the monoidal case, the obvious guess is to completely remove the combinator “braid” and all

rewrite rules involving it. In fact, Jay [57] gave a language suitable for closed monoidal categories in 1989; Ambler’s work is based on this.

## 2.5 Conclusions

In this paper we sketched how category theory can serve to clarify the analogies between physics, topology, logic and computation. Each field has its own concept of “thing” (object) and “process” (morphism)—and these things and processes are organized into categories that share many common features. To keep our task manageable, we focused on those features that are present in every closed symmetric monoidal category. Table 2.4, an expanded version of the Rosetta Stone, shows some of the analogies we found.

**Table 2.4** The Rosetta Stone (larger version)

Category Theory	Physics	Topology	Logic	Computation
Object $X$	Hilbert space $X$	Manifold $X$	Proposition $X$	Data type $X$
Morphism $f: X \rightarrow Y$	Operator $f: X \rightarrow Y$	Cobordism $f: X \rightarrow Y$	Proof $f: X \rightarrow Y$	Program $f: X \rightarrow Y$
Tensor product of objects: $X \otimes Y$	Hilbert space of joint system: $X \otimes Y$	Disjoint union of manifolds: $X \otimes Y$	Conjunction of propositions: $X \otimes Y$	Product of data types: $X \otimes Y$
Tensor product of morphisms: $f \otimes g$	Parallel processes: $f \otimes g$	Disjoint union of cobordisms: $f \otimes g$	Proofs carried out in parallel: $f \otimes g$	Programs executing in parallel: $f \otimes g$
Internal hom: $X \multimap Y$	Hilbert space of “anti- $X$ and $Y$ ”: $X^* \otimes Y$	Disjoint union of orientation-reversed $X$ and $Y$ : $X^* \otimes Y$	Conditional proposition: $X \multimap Y$	Function type: $X \multimap Y$

However, we only scratched the surface! There is much more to say about categories equipped with extra structure, and how we can use them to strengthen the ties between physics, topology, logic and computation—not to mention what happens when we go from categories to  $n$ -categories. But the real fun starts when we exploit these analogies to come up with new ideas and surprising connections. Here is an example.

In the late 1980s, Witten [109] realized that string theory was deeply connected to a 3d topological quantum field theory and thus the theory of knots and tangles [71]. This led to a huge explosion of work, which was ultimately distilled into a beautiful body of results focused on a certain class of compact braided monoidal categories called “modular tensor categories” [17, 107].

All this might seem of purely theoretical interest, were it not for the fact that superconducting thin films in magnetic fields seem to display an effect—the “fractional quantum Hall effect”—that can be nicely modelled with the help of such categories [102, 103]. In a nutshell, the idea is that excitations of these films can act

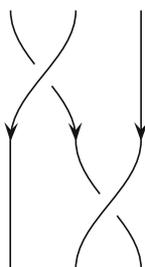
like particles, called ‘anyons’. When two anyons trade places, the result depends on how they go about it:



So, collections of anyons are described by objects in a braided monoidal category! The details depend on things like the strength of the magnetic field; the range of possibilities can be worked out with the help of modular tensor categories [82, 88].

So far this is all about physics and topology. Computation entered the game around 2000, when Freedman, Kitaev, Larsen and Wang [43–45] showed that certain systems of anyons could function as “universal quantum computers”. This means that, in principle, arbitrary computations can be carried out by moving anyons around. Doing this *in practice* will be far from easy. However, Microsoft has set up a research unit called Project Q attempting to do just this. After all, a working quantum computer could have huge practical consequences.

But regardless of whether topological quantum computation ever becomes practical, the implications are marvelous. A simple diagram like this:



can now be seen as a *quantum process*, a *tangle*, a *computation*—or an abstract morphism in any braided monoidal category! This is just the sort of thing one would hope for in a general science of systems and processes.

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## References

1. Abelson, H., Sussman, G.J., Sussman, J.: Structure and Interpretation of Computer Programs, MIT Press, Cambridge (1996). Available at <http://mitpress.mit.edu/sicp/> 152
2. Abramsky, S.: Abstract scalars, loops, and free traced and strongly compact closed categories. In: Proceedings of CALCO 2005, Lecture Notes in Computer Science 3629, Springer, Berlin, 2005, 1–31. Also available at <http://web.comlab.ox.ac.uk/oucl/work/samson.abramsky/calco05.pdf> 130, 132
3. Abramsky, S., Coecke, B.: A categorical semantics of quantum protocols. Available at [arXiv:quant-ph/0402130](http://arxiv.org/abs/quant-ph/0402130) 97, 130, 132
4. Abramsky, S., Duncan, R.: A categorical quantum logic, to appear in Mathematical Structures in Computer Science (2006). Also available as [arXiv:quant-ph/0512114](http://arxiv.org/abs/quant-ph/0512114) 130
5. Abramsky, S., Haghverdi, E., Scott, P.: Geometry of interaction and linear combinatory algebras, Math. Struct. Comput. Sci. **12**, 625–665 (2002). Also available at <http://citeseer.ist.psu.edu/491623.html>
6. Abramsky, S., Tzevelokos, N.: Introduction to categories and categorical logic, in this volume. Also available at <http://web.comlab.ox.ac.uk/people/Bob.Coecke/AbrNikos.pdf> 155
7. Ambler, S.: First order logic in symmetric monoidal closed categories, Ph.D. thesis, U. of Edinburgh, 1991. Available at <http://www.lfcs.inf.ed.ac.uk/reports/92/ECS-LFCS-92-194/> 155, 159
8. Atiyah, M.F.: Topological quantum field theories. Publ. Math. IHES Paris **68**, 175–186 (1989). 105
9. Atiyah, M.F.: The Geometry and Physics of Knots. Cambridge University Press, Cambridge (1990) 105
10. Baez, J.: An introduction to spin foam models of quantum gravity and *BF* theory. In: Gausterer, H., Grosse, H. (eds.) Geometry and Quantum Physics, pp. 25–93. Springer, Berlin (2000). Also available at [arXiv:gr-qc/9905087](http://arxiv.org/abs/gr-qc/9905087) 99
11. Baez, J.: Higher-dimensional algebra and Planck-scale physics. In: Callender, C., Huggett, N. (eds.) Physics Meets Philosophy at the Planck Length, pp. 177–195. Cambridge University Press, Cambridge (2001). Also available as [arXiv:gr-qc/9902017](http://arxiv.org/abs/gr-qc/9902017) 99
12. Baez, J.: Quantum quandaries: A category-theoretic perspective. In: French, S., Rickles, D., Saatsi, J. (eds.) Structural Foundations of Quantum Gravity, pp. 240–265. Oxford University Press, Oxford (2006). Also available as [arXiv:quant-ph/0404040](http://arxiv.org/abs/quant-ph/0404040) 99, 102, 132
13. Baez, J.: Classical versus quantum computation, U. C. Riverside seminar notes by D. Wise. Available at <http://math.ucr.edu/home/baez/qg-fall2006/> and <http://math.ucr.edu/home/baez/qg-winter2007> 167
14. Baez, J., Dolan, J.: Higher-dimensional algebra and topological quantum field theory. J. Math. Phys. **36**, 6073–6105 (1995). Also available as [arXiv:q-alg/9503002](http://arxiv.org/abs/q-alg/9503002) 99, 104, 119, 120, 132
15. Baez, J., Langford, L.: Higher-dimensional algebra IV: 2-tangles. Adv. Math. **180**, 705–764 (2003). Also available as [arXiv:q-alg/9703033](http://arxiv.org/abs/q-alg/9703033) 120, 132
16. Baez, J., Lauda, A.: A prehistory of *n*-categorical physics. In: Hans Halvorson (ed.) To Appear in Proceedings of Deep Beauty: Mathematical Innovation and the Search for an Underlying Intelligibility of the Quantum World, Princeton, October 3, 2007. Also available at <http://math.ucr.edu/home/baez/history.pdf> 120
17. Bakalov, B., Kirillov, A., Jr.: Lectures on Tensor Categories and Modular Functors, American Mathematical Society, Providence, Rhode Island, 2001. Preliminary version available at <http://www.math.sunysb.edu/~kirillov/tensor/tensor.html> 130, 166
18. Barendregt, H.: The Lambda Calculus, Its Syntax and Semantics. North-Holland, Amsterdam (1984) 149
19. Barr, M., Wells, C.: Toposes, Triples and Theories. Springer, Berlin (1983). Revised and corrected version available at <http://www.cwru.edu/artsci/math/wells/pub/ttt.html>
20. Bell, J.S.: On the Einstein-Podolsky-Rosen paradox, Physics **1**, 195–200 (1964) 115
21. Bell, J.L.: The Development of Categorical Logic. Available at <http://publish.uwo.ca/~jbell/catlogprime.pdf> 100

22. Benton, N., Bierman, G.M., de Paiva, V., Hyland, J.M.E.: Linear lambda-calculus and categorical models revisited, in *Computer Science Logic (CSL'92), Selected Papers, Lecture Notes in Computer Science* **702**, pp. 61–84. Springer, Berlin (1992). Also available at <http://citeseer.ist.psu.edu/benton92linear.html> [155](#)
23. Benton, N., Bierman, G., de Paiva, V., Hyland, M.: Term Assignment for Intuitionistic Linear Logic, Technical Report 262, University of Cambridge Computer Laboratory, August 1992. Also available at <http://citeseer.ist.psu.edu/1273.html>
24. Bierman, G.: On Intuitionistic Linear Logic, PhD Thesis, Cambridge University. Available at [http://research.microsoft.com/\\_gmb/Papers/thesis.pdf](http://research.microsoft.com/_gmb/Papers/thesis.pdf) [145](#)
25. Blute, R., Scott, P.: Category theory for linear logicians. In: Ehrhard, T., Girard, J.-Y., Ruet, P., Scott, P. (eds.) *Linear Logic in Computer Science*, pp. 3–64. Cambridge University Press, Cambridge (2004). Also available at [http://www.site.uottawa.ca/\\_phil/papers/catsurv.web.pdf](http://www.site.uottawa.ca/_phil/papers/catsurv.web.pdf) [145](#)
26. Burris, S.N., Sankappanavar, H.P.: *A Course in Universal Algebra*. Springer, Berlin (1981). Also available at [http://www.math.uwaterloo.ca/\\_snburris/htdocs/ualg.html](http://www.math.uwaterloo.ca/_snburris/htdocs/ualg.html) [145](#)
27. Chari, V., Pressley, A.: *A Guide to Quantum Groups*. Cambridge University Press, Cambridge (1995)
28. Cheng, E., Lauda, A.: *Higher-Dimensional Categories: An Illustrated Guidebook*. Available at [http://www.dpmms.cam.ac.uk/\\_elgc2/guidebook/](http://www.dpmms.cam.ac.uk/_elgc2/guidebook/)
29. Nielsen, M.A., Chuang, I.L.: *Quantum Computation and Quantum Information*. Cambridge University Press, Cambridge (2000) [97](#), [155](#)
30. Church, A.: An unsolvable problem of elementary number theory. *Am. J. Math.* **58**, 345–363 (1936) [149](#)
31. Coecke, B.: De-linearizing linearity: projective quantum axiomatics from strong compact closure. In: *Proceedings of the 3rd International Workshop on Quantum Programming Languages (QPL 2005)*, pp. 49–72. Elsevier (2007). Also available as arXiv:quantph/0506134 [130](#)
32. Coecke, B.: Kindergarten quantum mechanics. To appear in *Proceedings of QTRF-III*. Also available as arXiv:quant-ph/0510032 [130](#)
33. Coecke, B., Paquette, E.O.: POVMs and Naimark's theorem without sums. To appear in *Proceedings of the 4th International Workshop on Quantum Programming Languages (QPL 2006)*. Also available as arXiv:quant-ph/0608072 [130](#)
34. Coecke, B., Pavlovic, D.: Quantum measurements without sums. In: Chen, G., Kauffman, L., Lomonaco, S. (eds.) *To appear in the Mathematics of Quantum Computation and Technology*. Taylor and Francis. Also available as arXiv:quant-ph/0608035 [130](#)
35. Crole, R.L.: *Categories for Types*. Cambridge University Press, Cambridge (1993) [100](#), [145](#), [154](#)
36. Curry, H.B., Feys, R.: *Combinatory Logic*, vol. I. North-Holland, Amsterdam (1958) [97](#), [122](#), [159](#)
37. Curry, H.B., Findley, J.R., Selding, J.P.: *Combinatory Logic*, vol. II. North-Holland, Amsterdam (1972) [97](#), [122](#), [159](#)
38. Cvitanovic, P.: *Group Theory*. Princeton University Press, Princeton (2003). Available at <http://www.nbi.dk/GroupTheory/> [130](#)
39. Di Cosmo, R., Miller, D.: Linear logic, *Stanford Encyclopedia of Philosophy*. Available at <http://plato.stanford.edu/entries/logic-linear/> [145](#)
40. Dorca, M., van Tonder, A.: Quantum computation, categorical semantics and linear logic. Available as arXiv:quant-ph/0312174 [100](#)
41. Eilenberg, S., Kelly, G.M.: Closed categories. In: *Proceedings of the Conference on Categorical Algebra (La Jolla, 1965)*, pp. 421–562. Springer, Berlin (1966) [120](#)
42. Eilenberg, S., Mac Lane, S.: General theory of natural equivalences. *Trans. Am. Math. Soc.* **58**, 231–294 (1945)
43. Freedman, M., Kitaev, A., Larsen, M., Wang, Z.: Topological quantum computation. Available as arXiv:quant-ph/0101025 [167](#)
44. Freedman, M., Kitaev, A., Wang, Z.: Simulation of topological field theories by quantum computers. *Commun. Math. Phys.* **227**, 587–603 (2002). Also available as arXiv:quantph/0001071 [167](#)

45. Freedman, M., Kitaev, A., Wang, Z.: A modular functor which is universal for quantum computation. *Commun. Math. Phys.* **227**, 605–622 (2002). Also available as arXiv:quantph/0001108 [167](#)
46. Freyd, P., Yetter, D.: Braided compact monoidal categories with applications to low dimensional topology. *Adv. Math.* **77**, 156–182 (1989) [103](#), [125](#)
47. Girard, J.-Y.: Linear logic. *Theor. Comp. Sci.* **50**, 1–102 (1987). Also available at <http://iml.univ-mrs.fr/girard/linear.pdf> [97](#), [136](#), [137](#), [140](#), [145](#)
48. Girard, J.-Y., Lafont, Y., Taylor, P.: *Proofs and Types*. Cambridge University Press, Cambridge (1990). Also available at <http://www.monad.me.uk/stable/Proofs%2BTypes.html> [135](#)
49. Gödel, K.: Zur intuitionistischen Arithmetik und Zahlentheorie. *Ergebnisse eines mathematischen Kolloquiums* **4**, 34–38 (1933) [133](#)
50. Goldblatt, R.: *Topoi: The Categorical Analysis of Logic*. North-Holland, New York (1984). Also available at <http://cdl.library.cornell.edu/cgi-bin/cul.math/docviewer?did=Gold010> [100](#)
51. Gottesman, D., Chuang, I.L.: Quantum teleportation is a universal computational primitive. *Nature* **402**, 390–393 (1999). Also available as arXiv:quant-ph/9908010 [120](#), [128](#)
52. Hasegawa, M.: Logical predicates for intuitionistic linear type theories. In: Girard, J.-Y. (ed.) *Typed Lambda Calculi and Applications: 4th International Conference, TLCA '99*. Lecture Notes in Computer Science **1581**. Springer, Berlin (1999). Also available at <http://citeseer.ist.psu.edu/187161.html> [137](#), [145](#)
53. Heyting, A., Brouwer, L.E.J. (ed.): *Collected Works 1: Philosophy and Foundations of Mathematics*. Elsevier, Amsterdam (1975) [133](#)
54. Hilken, B.: Towards a proof theory of rewriting: The simply-typed  $2^*$ -calculus. *Theor. Comput. Sci.* **170**, 407–444 (1996) [154](#)
55. Hindley, J.R., Seldin, J.P.: *Lambda-Calculus and Combinators: An Introduction*. Cambridge University Press, Cambridge (2008) [149](#)
56. Howard, W.A.: The formulae-as-types notion of constructions. In: Seldin, J.P., Hindley, J.R. (eds.) *To Curry, H.B.: Essays on Combinatory Logic, Lambda Calculus and Formalism*, pp. 479–490. Academic Press, New York (1980) [97](#)
57. Jay, C.B.: Languages for monoidal categories. *J. Pure Appl. Alg.* **59**, 61–85 (1989) [166](#)
58. Jay, C.B.: The structure of free closed categories. *J. Pure Appl. Alg.* **66**, 271–285 (1990) [145](#)
59. Joyal, A., Street, R.: The geometry of tensor calculus I. *Adv. Math.* **88**, 55–113 (1991) [110](#)
60. Joyal, A., Street, R.: The geometry of tensor calculus II. Available at <http://www.math.mq.edu.au/street/GTCII.pdf> [110](#)
61. Joyal, A., Street, R.: Braided monoidal categories, Macquarie Math Reports 860081 (1986). Available at [http://rutherglen.ics.mq.edu.au/\\_street/JS86.pdf](http://rutherglen.ics.mq.edu.au/_street/JS86.pdf) [118](#)
62. Joyal, A., Street, R.: Braided tensor categories. *Adv. Math.* **102**, 20–78 (1993) [118](#)
63. Kaiser, D.: *Drawing Theories Apart: The Dispersion of Feynman Diagrams in Postwar Physics*. University of Chicago Press, Chicago (2005) [96](#)
64. Kassel, C.: *Quantum Groups*. Springer, Berlin (1995) [103](#), [130](#)
65. Kauffman, L.H.: *Knots and Physics*. World Scientific, Singapore (1991) [130](#)
66. Kauffman, L.H., Lins, S.: *Temperley–Lieb Recoupling Theory and Invariants of 3-Manifolds*. Princeton University Press, Princeton (1994) [130](#)
67. Kelly, G.M., Mac Lane, S.: Coherence in closed categories. *J. Pure Appl. Alg.* **1**, 97–140, 219 (1971) [145](#)
68. Kelly, G.M., Laplaza, M.L.: Coherence for compact closed categories. *J. Pure Appl. Alg.* **19**, 193–213 (1980) [145](#)
69. Kleene, S.:  $*$ -definability and recursiveness. *Duke Math. J.* **2**, 340–353 (1936) [149](#)
70. Kock, J.: *Frobenius Algebras and 2D Topological Quantum Field Theories*, London Mathematical Society Student Texts **59**. Cambridge University Press, Cambridge (2004) [130](#)
71. Kohno, T. (ed.): *New Developments in the Theory of Knots*. World Scientific, Singapore (1990) [166](#)
72. Lambek, J.: From  $*$ -calculus to cartesian closed categories. In: Seldin, J.P., Hindley, J.R. (eds.) *To Curry, H.B.: Essays on Combinatory Logic, Lambda Calculus and Formalism*, pp. 375–402. Academic Press, New York (1980) [154](#)

73. Lambek, J., Scott, P.J.: Introduction to Higher-order Categorical Logic. Cambridge University Press, Cambridge (1986) [146](#), [149](#), [154](#), [158](#)
74. Landin, P.: A correspondence between ALGOL 60 and Church's lambda-notation. *Commun. ACM* **8**, 89–101, 158–165 (1965) [152](#)
75. Lawvere, F.W.: Functorial Semantics of Algebraic Theories, Ph.D. Dissertation, Columbia University (1963). Also available at <http://www.tac.mta.ca/tac/reprints/articles/5/tr5abs.html> [105](#), [145](#)
76. Leinster, T.: A survey of definitions of  $n$ -category. *Theor. Appl. Cat.* **10**, 1–70 (2002). Also available as arXiv:math/0107188
77. Mac Lane, S.: Natural associativity and commutativity. *Rice Univ. Stud.* **49** 28–46 (1963) [106](#), [109](#), [119](#), [147](#)
78. Mac Lane, S.: Categories for the Working Mathematician. Springer, Berlin (1998) [100](#), [131](#)
79. Melliès, P.: Axiomatic rewriting theory I: A diagrammatic standardisation theorem. In: Processes, Terms and Cycles: Steps on the Road to Infinity, Lecture Notes in Computer Science **3838**, pp. 554–638. Springer, New York (2005). Also available at <http://www.pps.jussieu.fr/mellies/papers/jwkfestschrift.pdf> [154](#)
80. McCarthy, J.: Recursive functions of symbolic expressions and their computation by machine, Part I. *Comm. ACM* **4**, 184–195 (1960). Also available at <http://www.formal.stanford.edu/jmc/recursive.html> [152](#)
81. McLarty, C.: Elementary Categories, Elementary Toposes. Clarendon Press, Oxford (1995) [100](#)
82. Moore, G., Read, N.: Nonabelions in the fractional quantum Hall effect. *Nucl. Phys. B* **360**, 362–396 (1991) [167](#)
83. Moschovakis, J.: Intuitionistic logic, *Stanford Encyclopedia of Philosophy*. Available at <http://plato.stanford.edu/entries/logic-intuitionistic/> [146](#)
84. Penrose, R.: Applications of negative dimensional tensors. In: Welsh, D. (ed.) *Combinatorial Mathematics and Its Applications*, pp. 221–244. Academic Press, New York (1971) [96](#)
85. Penrose, R.: Angular momentum: An approach to combinatorial space-time. In: Bastin, T. (ed.) *Quantum Theory and Beyond*, pp. 151–180. Cambridge University Press, Cambridge (1971) [96](#)
86. Penrose, R.: On the nature of quantum geometry. In: Klauder J. (ed.) *Magic Without Magic*, pp. 333–354. Freeman, San Francisco (1972) [96](#)
87. Penrose, R.: Combinatorial quantum theory and quantized directions. In: Hughston, L., Ward, R. (eds.) *Advances in Twistor Theory*, Pitman Advanced Publishing Program, pp. 301–317 (1979) [96](#)
88. Rowell, E., Stong, R., Wang, Z.: On classification of modular tensor categories, available as arXiv:0712.1377 [167](#)
89. Scott, P.: Some aspects of categories in computer science. In: Hazewinkel, M. (ed.) *Handbook of Algebra*, vol. 2, Elsevier, Amsterdam (2000). Also available at <http://www.site.uottawa.ca/phil/papers/handbook.ps> [154](#)
90. Sawin, S.: Links, quantum groups and TQFTs. *Bull. Am. Math. Soc.* **33**, 413–445 (1996). Also available as arXiv:q-alg/9506002 [102](#), [103](#)
91. Schalk, A.: What is a categorical model for linear logic? Available at [http://www.cs.man.ac.uk/\\_schalk/notes/llmodel.pdf](http://www.cs.man.ac.uk/_schalk/notes/llmodel.pdf) [137](#), [145](#)
92. Schönfinkel, M.: Über die Bausteine der mathematischen Logik. *Math. Ann.* **92**, 305–316 (1924). Also available as On the building blocks of mathematical logic. *Trans. Bauer-Mengelberg, S.*: In: van Heijenoort, J. (ed.) *A Source Book in Mathematical Logic, 1879–1931*, pp. 355–366. Harvard University Press, Cambridge, Massachusetts (1967) [159](#)
93. Seely, R.A.G.: Weak adjointness in proof theory, *Applications of Sheaves*, Lecture Notes in Mathematics **753**, pp. 697–701. Springer, Berlin. Also available at <http://www.math.mcgill.ca/rags/WkAdj/adj.pdf> [143](#), [154](#)
94. Seely, R.A.G.: Modeling computations: A 2-categorical framework. In: *Proceedings of the Symposium Logic Computer Science 1987*, Computer Society of the IEEE, pp. 65–71. Also available at <http://www.math.mcgill.ca/rags/WkAdj/LICS.pdf> [143](#), [154](#)
95. Segal, G.: The definition of a conformal field theory. In: Tillmann, U.L. (ed.) *Topology, Geometry and Quantum Field Theory: Proceedings of the 2002 Oxford Symposium in Honour of the 60th Birthday of Graeme Segal*. Cambridge University Press, Cambridge (2004) [105](#)

96. Selinger, P.: Lecture notes on the lambda calculus. Available at <http://www.mscs.dal.ca/~selinger/papers/#lambdanotes> 149
97. Selinger, P.: Dagger compact closed categories and completely positive maps. In: Proceedings of the 3rd International Workshop on Quantum Programming Languages (QPL 2005), pp. 139–163. Elsevier, Amsterdam (2007). Also available at <http://www.mscs.dal.ca/~selinger/papers/#dagger> 132
98. Selinger, P., Valiron, B.: A lambda calculus for quantum computation with classical control. *Math. Struct. Comp. Sci.* **13**, 527–552 (2006). Also available at <http://www.mathstat.dal.ca/~selinger/papers/#qlambda> 155
99. Shum, M.-C.: Tortile tensor categories. *J. Pure Appl. Alg.* **93**, 57–110 (1994) 103, 104
100. Soloviev, S.: Proof of a conjecture of S. Mac Lane. *Ann. Pure Appl. Logic* **90**, 101–162 (1997) 145
101. Smolin, L.: The future of spin networks. In: Hugget, S., Tod, P., Mason, L.J. (eds.) *The Geometric Universe: Science, Geometry, and the Work of Roger Penrose*. Oxford University Press, Oxford (1998). Also available as arXiv:gr-qc/9702030 130
102. Stern, A.: Anyons and the quantum Hall effect – A pedagogical review. *Ann. Phys.* **323**, 204–249 (2008). Available as arXiv:0711.4697 166
103. Stone, M. (ed.): *Quantum Hall Effect*. World Scientific, Singapore (1992) 166
104. Szabo, M.E. (ed.): *Collected Papers of Gerhard Gentzen*. North-Holland, Amsterdam (1969) 133, 135
105. Trimble, T.: *Linear Logic, Bimodules, and Full Coherence for Autonomous Categories*, Ph.D. thesis, Rutgers University (1994) 145
106. Troelstra, A.S.: *Lectures on Linear Logic*, Center for the Study of Language and Information. Stanford, California (1992) 145
107. Turaev, V.G.: *Quantum Invariants of Knots and 3-Manifolds*. de Gruyter, Berlin (1994) 103, 130, 166
108. van Tonder, A.: A lambda calculus for quantum computation. *SIAM J. Comput.* **33**, 1109–1135 (2004). Also available as arXiv:quant-ph/0307150 155
109. Witten, E.: Quantum field theory and the Jones polynomial. *Commun. Math. Phys.* **121**, 351–399 (1989) 166
110. Wootters, W.K., Zurek, W.H.: A single quantum cannot be cloned. *Nature* **299**, 802–803 (1982) 115, 155
111. Yetter, D.N.: *Functorial Knot Theory: Categories of Tangles, Coherence, Categorical Deformations, and Topological Invariants*. World Scientific, Singapore (2001) 103, 110