# Lecture 8: The Curry-Howard-Lambek correspondence for classical computation

**Summary.**
(1) The Curry-Howard-Lambek correspondence: from logic to categories and back.

(2) The Curry-Howard-Lambek correspondence: from programs to categories and back.

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**Overview.**

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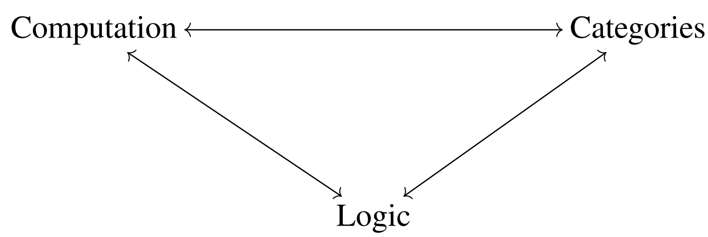
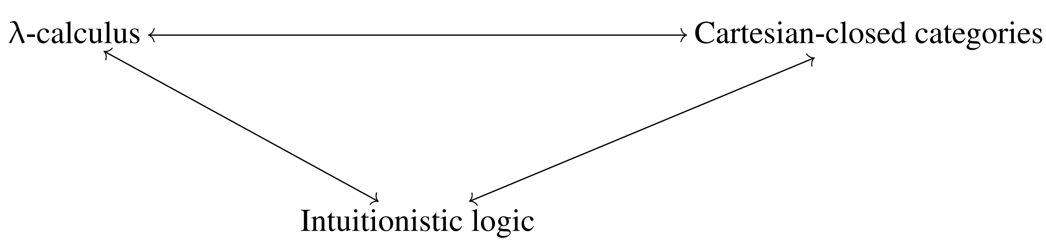
The general triangle




is instantiated to




A previous lecture already discussed the link between (intuitionistic) logic and (simply-typed) λ-calculus under the *motto*

Formulas-as-Types and Proofs-as-Programs

It was emphasized that exploring the computational content of proofs is, indeed, fully aligned with the constructive (BHK) interpretation of intuitionistic logic under which, for example, a proof of $A \wedge B$ is a pair of proofs of both $A$ and $B$, and a proof of $A \longrightarrow B$ is a procedure to transform any proof of $A$ into a proof of $B$. We turn now to the links that both logic and computation keep with the mathematical structures which provide their semantical models, i.e with *categories*.

Given a Cartesian-closed category (CCC) $\mathcal{C}$, the Lambek's part of the diagram identifies

Formulas-as-Objects and Proofs-as-Arrows

Recall the basic structure of a CCC:

- *Products*: $A \times B$, with projections $\pi_1, \pi_2$ and a split arrow $\langle f, g \rangle : C \longrightarrow A \times B$ defined as a universal property from $f : C \longrightarrow A$ and $g : C \longrightarrow g$ . Thus a proof of $A$ from assumptions $B_1$ to $B_n$ corresponds to a morphism

  $$f : B_1 \times \cdots \times B_n \longrightarrow A$$

  The product construction is functorial: $f \times g = \langle f \cdot \pi_1, g \cdot \pi_2 \rangle$.

- *Exponentials*: $B^A$, given through the natural isomorphism between

  $$f : A \times B \longrightarrow C \quad \Leftrightarrow \quad \overline{f} : A \longrightarrow C^B$$

  expressed through another universal property

  $$k = \overline{f} \quad \Leftrightarrow \quad f = ev \cdot (k \times id)$$

  The exponential construction is also functorial: $f^C \;=\; f \cdot \_$.

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**The link to logic.**

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Formulas in intuitionistic logic correspond objects in $\mathcal{C}$; proofs correspond to morphisms in $\mathcal{C}$. The correspondence is as follows:

| Intuitionistic logic | CCC |
|---|---|
| $\dfrac{}{\Gamma, x : A \vdash A}$ | $\dfrac{}{\pi_2 : \Gamma \times A \longrightarrow A}$ |
| $\dfrac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$ | $\dfrac{f : \Gamma \longrightarrow A \quad g : \Gamma \longrightarrow B}{\langle f, g \rangle : \Gamma \longrightarrow A \times B}$ |
| $\dfrac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}$ | $\dfrac{f : \Gamma \longrightarrow A \times B}{\pi_1 \cdot f : \Gamma \longrightarrow A}$ |
| $\dfrac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}$ | $\dfrac{f : \Gamma \longrightarrow A \times B}{\pi_2 \cdot f : \Gamma \longrightarrow B}$ |
| $\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \longrightarrow B}$ | $\dfrac{f : \Gamma \times A \longrightarrow B}{\overline{f} : \Gamma \longrightarrow B^A}$ |
| $\dfrac{\Gamma \vdash A \longrightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B}$ | $\dfrac{f : \Gamma \longrightarrow B^A \quad g : \Gamma \longrightarrow A}{ev_{A,B} \cdot \langle f, g \rangle : \Gamma \longrightarrow B}$ |

**Exercise** 1

Extend the CHL correspondence to capture the propositional intuitionistic logic is enriched with disjunction, i.e. connectives $\vee$ and $\perp$.

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**The link to computation.**

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Types-as-Objects and Terms-as-Arrows

Types in the simply-typed $\lambda$-calculus correspond objects in a CCC $\mathcal{C}$. Terms, on the other hand, correspond to morphisms in $\mathcal{C}$. Moreover, the $\beta, \eta$-reduction is suitably derived from the axioms of a CCC. The correspondence is captured by a semantic function which translates each term

$$x_1 : A_1, \cdots, x_n : A_n \vdash u : B$$

into an arrow in $\mathcal{C}$:

$$[\![u]\!] : [\![A_1]\!] \times \cdots \times [\![A_n]\!] \longrightarrow [\![B]\!]$$

The correspondence is defined recursively on types by

$$\begin{aligned} [\![A \times B]\!] &\mathrel{\widehat{=}} [\![A]\!] \times [\![B]\!] \\ [\![A \longrightarrow B]\!] &\mathrel{\widehat{=}} [\![B]\!]^{[\![A]\!]} \end{aligned}$$

assuming a set of distinguished objects in $\mathcal{C}$ as semantic domains for the basic types.

Similarly, for terms,

$$\frac{}{[\![\Gamma, x : A \vdash x : A]\!] \;\widehat{=}\; \pi_2 : [\![\Gamma]\!] \times [\![A]\!] \longrightarrow [\![A]\!]}$$

$$\frac{[\![\Gamma \vdash u : A \times B]\!] \;=\; f : [\![\Gamma]\!] \longrightarrow [\![A]\!] \times [\![B]\!]}{[\![\Gamma \vdash \pi_1\, u : A]\!] \;\widehat{=}\; \pi_1 \cdot f : [\![\Gamma]\!] \longrightarrow [\![A]\!]}$$

$$\frac{[\![\Gamma \vdash u : A]\!] \;=\; f : [\![\Gamma]\!] \longrightarrow [\![A]\!] \qquad [\![\Gamma \;\vdash\; v : B]\!] \;=\; g : [\![\Gamma]\!] \longrightarrow [\![B]\!]}{[\![\Gamma \vdash \langle u, v\rangle : A \times B]\!] \;\widehat{=}\; \langle f, g\rangle : [\![\Gamma]\!] \longrightarrow [\![A]\!] \times [\![B]\!]}$$

$$\frac{[\![\Gamma, x : A \vdash u : B]\!] \;=\; f : [\![\Gamma]\!] \times [\![A]\!] \longrightarrow [\![B]\!]}{[\![\Gamma \vdash \lambda x\,.\,u : A \longrightarrow B]\!] \;\widehat{=}\; \overline{f} : [\![\Gamma]\!] \longrightarrow [\![B]\!]^{[\![A]\!]}}$$

$$\frac{[\![\Gamma \vdash u : A \longrightarrow B]\!] \;=\; f \qquad [\![\Gamma \vdash v : A]\!] \;=\; g}{[\![\Gamma \vdash u\, v : B]\!] \;\widehat{=}\; ev \cdot \langle f, g\rangle : [\![A]\!] \longrightarrow [\![B]\!]}$$

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**Soundness of $[\![-]\!]$.**

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Soundness of the translation of simply-typed $\lambda$-calculus to a CCC means that $\beta, \eta$-equivalence, which equates terms that are derived one from the other through the rules of $\beta, \eta$-reduction, correspond to semantic equality, i.e.

$$\boxed{u =_{\beta,\eta} v \;\;\Rightarrow\;\; [\![u]\!] = [\![v]\!]}$$

Let $\Gamma = x_1 : A_1 \cdots A_n$. Given types terms $\Gamma \vdash u : A$ and, for all $1 \leq i \leq n$, $\Gamma \vdash u_i\ A$,

$$[\![u[x_1 := u_1, \cdots, x_n := u_n]]\!] \;=\; [\![u]\!] \cdot \langle [\![u_1]\!] \;\cdots\;, [\![u_n]\!]\rangle$$

This statement, known as the *substitution lemma*, is proved by induction on the structure of terms. The base case is that of variables: $x_i$. Actually,

$$[\![x_i[\mathbf{x} := \mathbf{u}]]\!] \;=\; [\![u_i]\!] \;=\; \pi_i \cdot \langle [\![u_1]\!], \cdots, [\![u_k]\!]\rangle \;=\; [\![x_i]\!] \cdot \langle [\![u_1]\!], \cdots, [\![u_k]\!]\rangle$$

For the inductive process, consider, for example, $\lambda x \,.\, u$. Thus,

$$\begin{array}{cl}
 & [\![\lambda x \,.\, u[\mathbf{x} := \mathbf{v}]]\!] \\
= & \{\ \text{substitution}\ \} \\
 & [\![\lambda x \,.\, u[\mathbf{x}, x := \mathbf{v}, x]]\!] \\
= & \{\ [\![-]\!]\ \text{definition}\ \} \\
 & \overline{[\![u[\mathbf{x}, x := \mathbf{v}, x]]\!]} \\
= & \{\ \text{induction hypothesis}\ \} \\
 & \overline{[\![u]\!] \cdot (\langle \mathbf{v}\rangle \times id)} \\
= & \{\ \text{fusion law for exponentials: } \overline{f} \cdot g \;=\; \overline{f \cdot (g \times id)}\ \} \\
 & \overline{[\![u]\!]} \cdot \langle \mathbf{v}\rangle \\
= & \{\ [\![-]\!]\ \text{definition}\ \} \\
 & [\![\lambda x \,.\, u]\!] \cdot \langle \mathbf{v}\rangle
\end{array}$$

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**Exercise** 2

Complete the proof of the *substitution lemma* above for the remaining cases.

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To establish soundness of the semantic interpretation $[\![\ ]\!]$, all we need to show is that the interpretation of both sides of a $\beta, \eta$-reduction corresponds to a valid equation in a CCC. The substitution lemma is an important tool in this proof.

Let us start with $\beta$-conversion, considering the interpretation of

$$(\lambda x \,.\, u)v \;=_\beta\; u[x := v]$$

$$
\begin{array}{cl}
 & [\![(\lambda x \,.\, u)v]\!] \\
= & \{\ [\![-]\!] \text{ definition } \} \\
 & ev \cdot \langle \overline{[\![u]\!]}, [\![v]\!] \rangle \\
= & \{\ \times\text{-absorption law} \} \\
 & ev \cdot (\overline{[\![u]\!]} \times id) \cdot \langle id, [\![v]\!] \rangle \\
= & \{ \text{currying definition} \} \\
 & [\![u]\!] \cdot \langle id, [\![v]\!] \rangle \\
= & \{ \text{substitution lemma} \} \\
 & [\![u[\mathbf{x}, v := \mathbf{x}, x]]\!]
\end{array}
$$

**Exercise** 3

Verify the second $=_\beta$-conversion

$$\pi_1 \langle u, v \rangle \ = \ u \qquad \text{and} \qquad \pi_2 \langle u, v \rangle \ = \ v$$

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**Exercise** 4

Verify the two $=_\eta$-conversions

$$u \ = \ \lambda x \,.\, u\, x$$

and

$$u \ = \ \langle \pi_1\, u, \pi_2\, u \rangle$$

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**Completeness of $[\![-]\!]$.**

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To show completeness one has to come up with a concrete CCC, $\Lambda$, in which equalities between arrows correspond to $\beta, \eta$-conversions between terms, i.e.

$$\boxed{u =_{\beta,\eta} v \ \Leftarrow \ [\![u]\!] = [\![v]\!]}$$

where $[\![-]\!]$ is an interpretation of $\lambda$-terms in $\Lambda$.

The category $\Lambda$ has an object $\hat{A}$ for each type $A$ in the $\lambda$-calculus, plus a final object $\mathbf{1}$. An arrow from $\hat{A}$ to $\hat{B}$ is an equivalence class of the following relation defined on variable-term pairs:

$$(x, u) \approx (y, v) \quad \text{iff} \quad x : A \vdash u : B \ \text{ and } \ y : A \vdash v : B \ \text{ and } \ u =_{\beta,\eta} v[y := x]$$

which extends to pairs $(*, u)$, where $*$ represents the single inhabitant of $\mathbf{1}$, as follows:

$$(*, u) \approx (*, v) \quad \text{iff} \quad \vdash u : B \;\text{ and }\; \vdash v : B \;\text{ and }\; u =_{\beta,\eta} v$$

As usual, the equivalence class $[(x, u)]$, for the element $(x, u)$, is the set $\{(y, v) \mid (x, u) \approx (y, v)\}$. Thus, the homsets of $\Lambda$ are as follows:

$$\begin{aligned} \Lambda\,[\hat{A}, \hat{B}] &= \{[(x, u)] \mid x : A \vdash u : B\} \\ \Lambda\,[\mathbf{1}, \hat{B}] &= \{[(*, u)] \mid \;\vdash u : B\} \\ \Lambda\,[\hat{A}, \mathbf{1}] &= \{!_{\hat{A}}\} \end{aligned}$$

**Exercise** 5

In $\Lambda$ define,

- Identities: $id_{\hat{A}} \;\widehat{=}\; [(x, x)]$ and $id_{\mathbf{1}} \;\widehat{=}\; !_{\mathbf{1}}$
- Composition:

$$\begin{aligned} [(x, u)] \cdot [(y, v)] &\;\widehat{=}\; [(y, u[x := v])] \\ [(x, u)] \cdot [(*, v)] &\;\widehat{=}\; [(*, u[x := v])] \\ [(*, u)] \cdot !_Z &\;\widehat{=}\; \begin{cases} [(y, u)] & \Leftarrow Z = \hat{A} \\ [(*, u)] & \Leftarrow Z = \mathbf{1} \end{cases} \\ !_W \cdot h &\;\widehat{=}\; !_Z \quad \text{for } h : Z \longrightarrow W \end{aligned}$$

Prove that $\Lambda$ is a category.

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The category $\Lambda$ has finite products and exponentials, and provides what is called a *term* (i.e. built on topo of the syntax) model for the simply-typed $\lambda$-calculus (see, e.g. [1] for proofs).

# References

[1] S. Abramsky and N. Tzevelekos. Introduction to categories and categorical logic. In B. Coecke, editor, *New Structures for Physics*, pages 3–94. Springer Lecture Notes on Physics (813), 2011.