Lecture 7: String diagrams and dagger categories - II

Summary.
(1) Unitary and positive processes. Projectors.
(2) Expressing quantum phenomena in string diagrams.
(3) Compact closed dagger categories.

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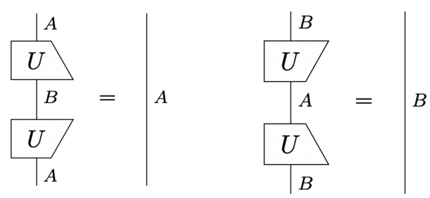
Unitary processes.

A process $U : A \longrightarrow B$ is unitary if $U^\dagger$ is its inverse, i.e. $U^\dagger \cdot U = id_A$ and $U \cdot U^\dagger = id_B$ Unitary processes are the ones that preserve the measure of commonality given by the inner product[1].



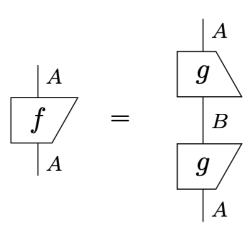
(isometry)

Exercise 1

Show that a unitary U preserves the inner product.

Positive processes.

A process $f : A \longrightarrow A$ is positive if there exists another process $g : A \longrightarrow B$ such that $f = g^\dagger \cdot g$, i.e.

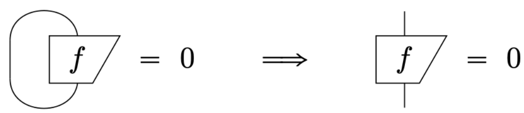


[1]Pictures are taken from Coecke and Kissinger book, *Picturing Quantum processes*, CUP, 2017.

The definition entails that positive processes are self-adjoint as they are invariant under vertical reflection. Note that the scalar representing the inner product of a state with itself is positive in this sense, which explains the qualifier *positive* when one requires the inner products to be *positive definite*, i.e. $\langle\phi|\phi\rangle = 0 \Leftrightarrow |\phi\rangle = 0$.

Exercise 2

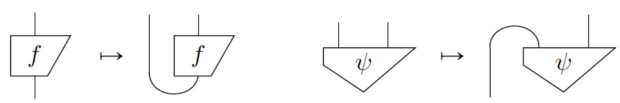
Show that if f is a positive process, $\mathrm{Tr}(f) = 0 \Rightarrow f = 0$, i.e.



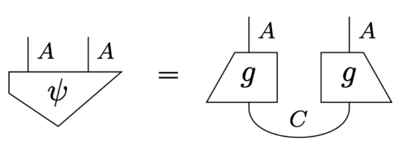
Exercise 3

In linear algebra f is positive if, for every ϕ, $\langle\phi|f|\phi\rangle > 0$. Relate this formulation to the definition just given.

In the previous lecture we have noted that string diagrams express a duality (i.e. a bijective correspondence) between processes and bipartite states, cf.



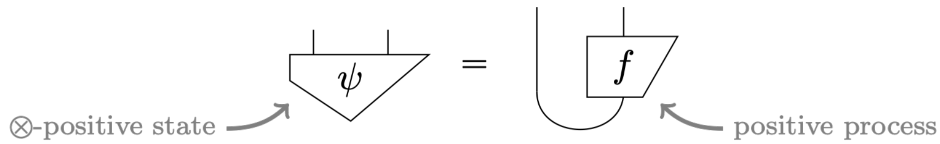
The state corresponding, under such a duality, to a positive process carried itself a positive structure in the horizontal dimension defined as follows: a bipartite state is $\otimes$-positive if there exists a process g such that



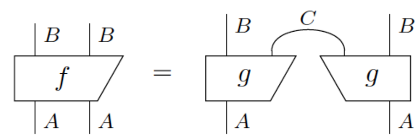
Thus,

Exercise 4

Verify the statement above, depicted as follows

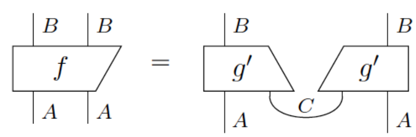


The definition extends to processes: f is ⊗-positive if there exists a process g such that



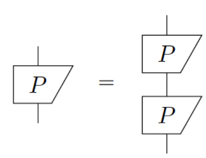
Exercise 5

Show this is equivalent to the existence of a process g' such that



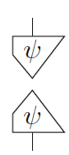
Projectors.

A process P positive and idempotent, i.e. such that



is called a *projector*.

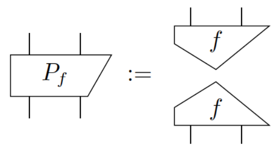
Any normalised state ψ yields a projector $|\psi\rangle\langle\psi|$ depicted as



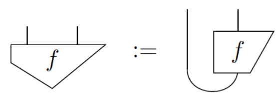
$\boxed{\textbf{Exercise } 6}$

Show this construction yields a positive and idempotent process.

In general, resorting to the duality between processes and bipartite states, one may define the notion of a *separable* projector as follows: A process $\mathsf{f} : \mathsf{A} \longrightarrow \mathsf{A}$ yields a separable projector via



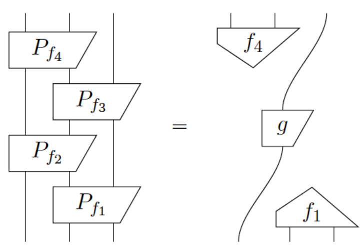
where state



is normalised. Note that a separable projector in linear algebra is exactly one that projects onto a one-dimensional vectorial space.

$\boxed{\textbf{Exercise } 7}$

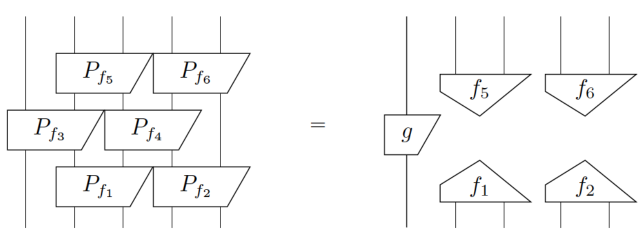
Show that



where $\mathsf{g} = \mathsf{f}_3 \cdot \overline{\mathsf{f}_4} \cdot \mathsf{f}_2^{\mathsf{T}} \cdot \mathsf{f}_3^{\dagger} \cdot \mathsf{f}_1 \cdot \mathsf{f}_1 \cdot \overline{\mathsf{f}_2}$

$\boxed{\textbf{Exercise } 8}$

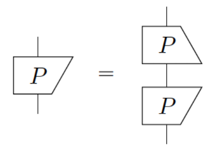
Show that



where $\mathsf{g} = \mathsf{f}_3^{\mathsf{T}} \cdot \mathsf{f}_5^{\dagger} \cdot \mathsf{f}_4^{\mathsf{T}} \cdot \mathsf{f}_6^{\dagger} \cdot \mathsf{f}_2 \cdot \overline{\mathsf{f}_4} \cdot \mathsf{f}_1 \cdot \overline{\mathsf{f}_3}$

Exercise 9

Show that one may define a projector, alternatively, as a self-adjoint idempotent or as a process P satisfying



Expressing quantum phenomena in string diagrams.

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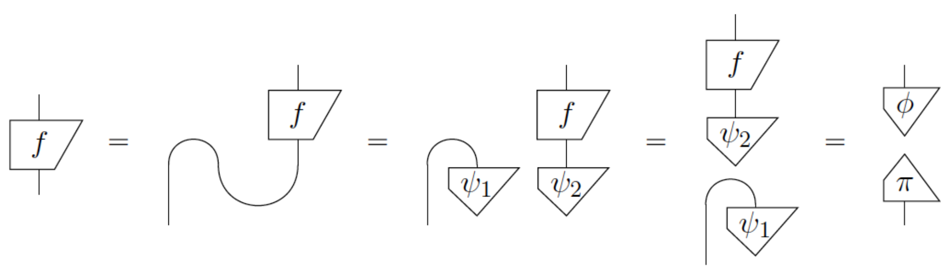
1. Non-separable states exist.

In a theory described by string diagrams, if all bipartite states are $\otimes$-separable, then all processes will be $\cdot$-separable, therefore making the theory trivial.

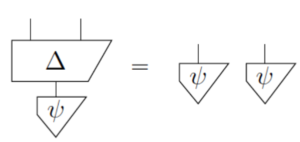
Proof.



for state $\phi = f \cdot \psi_2$ and effect $\pi = (\psi_1)^{\mathsf{T}}$. The second step assumes, by assumption, that cup is $\otimes$-separable.

2. The non-cloning theorem.

Let us define a *cloning process* Δ as one that makes two copies of its input state[2]

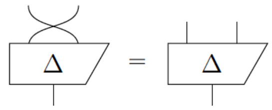


(1)

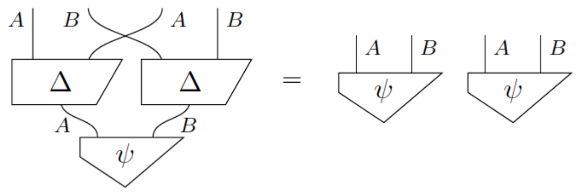
We formulate three reasonable assumptions on such a process:

[2]Note that in quantum information a *cloning process* is usually defined as a two inputs process whose second input gets overwritten by the first one. Our version captures the same phenomenon in a somehow less constrained way.

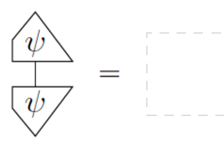
A (swapping does not affect cloning)



B (a composite is clones by cloning each of its components)

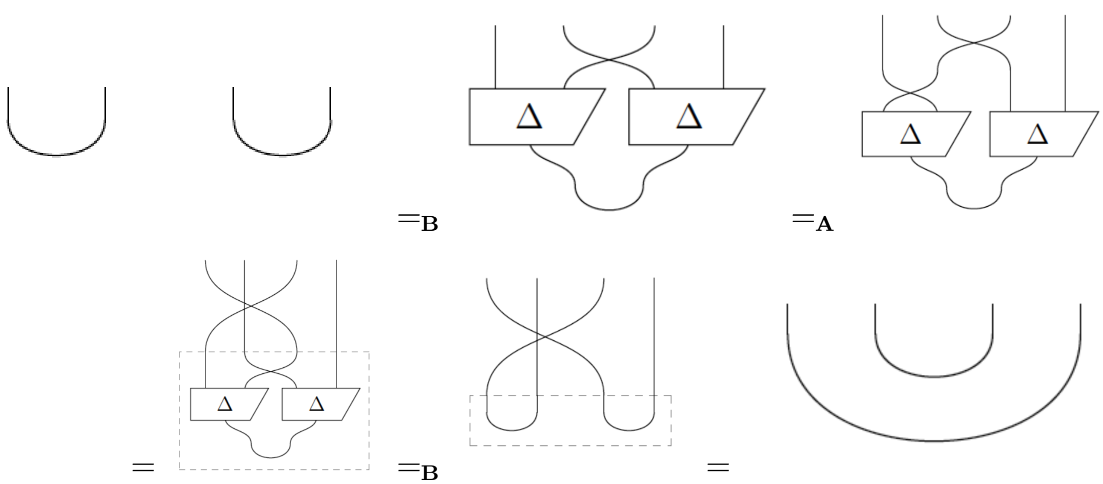


C (the process theory contains at least a normalised state)

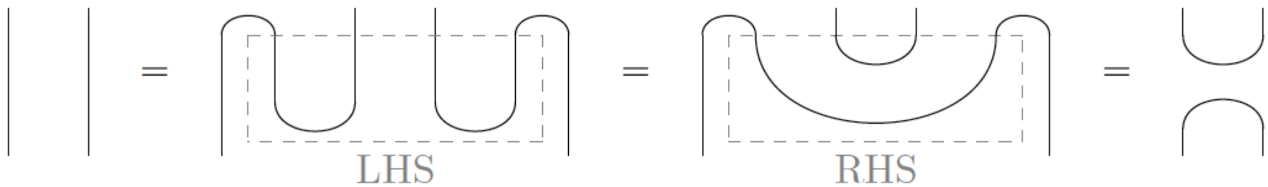


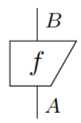
The *no-go* theorem is as follows: If a process theory described by string diagrams contains a cloning process for a type A, then every process with input A must be ·-separable.

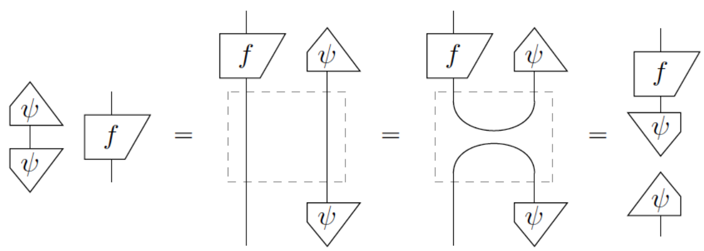
Proof.



where all wires are of type A. Converting outputs into inputs in both sides of the equation above yields



Then, by assumption **C**,  equals



The non-cloning theorem is folklore in quantum information. But what happens in the theory of relations? A cloning function is easily realised: $\Delta(a) = (a, a)$. Denoting by $\underline{x} : \mathbf{1} \longrightarrow A$ the constant function that always returns x, equation (1) defining a cloning process instantiates as follows:

$$\Delta(\underline{a}) \;=\; \underline{\Delta(a)} \;=\; (a, a) \;=\; \underline{a} \times \underline{a}$$

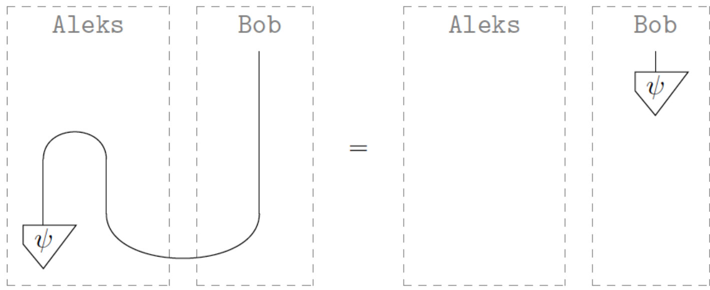
which is obviously true. Consider now a cloning relation $\Delta = \{(a, (a, a)) \mid a \in A\}$. Equation (1) now reads

$$\{(*, (a, a,)) \mid a \in A\} = \{(*, a) \mid a \in A\} \times \{(*, a) \mid a \in A\}$$

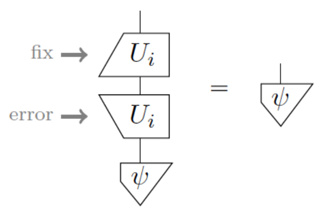
which is no longer true: the right hand side includes pairs $((*, a), (*, a))$ which are in bijective correspondence with pairs $(*, (a, a))$ in the left hand side, but also e.g. $((*, a), (*, b))$ for $a \neq b$. Note that in both process theories $\otimes$ is Cartesian product $\times$, but in the theory of relations this is not a categorical product.

3. A first version of teleportation.

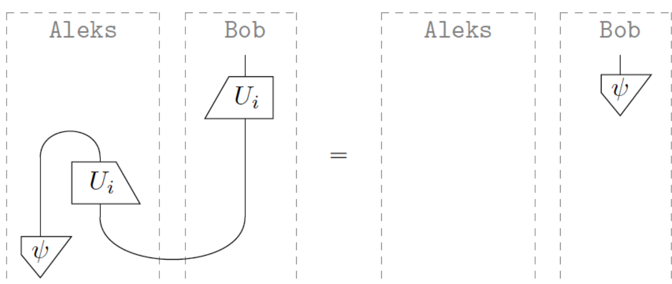
Assume Aleks possesses a state to be transmitted to Bob, with whom he shares a cup state. A solution may be



However, effects arise (to discuss later) as the result of a (quantum) *measurement*; thus Aleks might not get the cap itself, but the cap affected by some non-deterministic error from a given set of possible errors. Then Aleks needs to inform Bob of the error, i.e. to send a single index i so that Bob can choose the right error-corrector. Actually, assuming each U_i to be unitary, one has



leading to



Example: Teleportation in the theory of relations

The shared cup represents a pair of envelops, one for Aleks another for Bob, which inside have either a 0 or a 1. They do not know which bit is it, but they do know the bit is the same in both envelops. Formally, the shared cup represents this fact through the following relation

$$\cup = \{(*, (0,0)), (*, (1,1))\}$$

Aleks informs if the bit stored in his envelope is equal or different of his own bit ψ, which corresponds to the following effects, respectively:

$$\mathsf{M}_0 \;=\; \{((0,0),*),((1,1),*\} \qquad \mathsf{M}_1 \;=\; \{((0,1),*),((1,0),*)\}$$

From this information Bob may conclude if Alexs bit is the one in his own envelop or its complement. The correcting processes are, respectively,

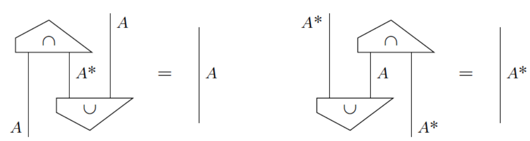
$$\mathsf{U}_0(x) = x \qquad \mathsf{U}_1(x) = 1 - x$$

Int the theory of relations this corresponds to what is known as a *one-time pad encryption*: Aleks sends public data — his bit encrypted by the parity measurement. Bob receives private data (after the right correction). A shared encryption key is used. In quantum teleportation Aleks sends classical data, Bob receives quantum data, using a shared quantum state.

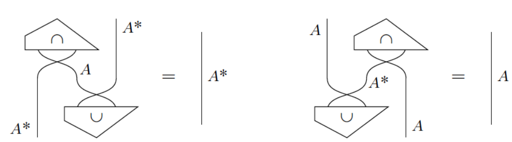
Dual objects.

String diagrams are sound and complete for *dagger compact closed categories.* These categories assume that each type A has a *dual*, A^* to which a cup state and a cap effect

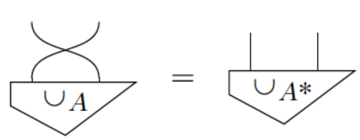
are associated and satisfy



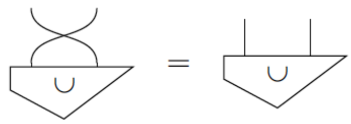
which, just by deformation, also yields



So, $(\mathsf{A}^*)^* \;=\; \mathsf{A}$ and, thus



When types are self-dual, i.e. $\mathsf{A} = \mathsf{A}^*$, as we have considered before, one gets two ways to define a cup for A, boiling down to the familiar equation



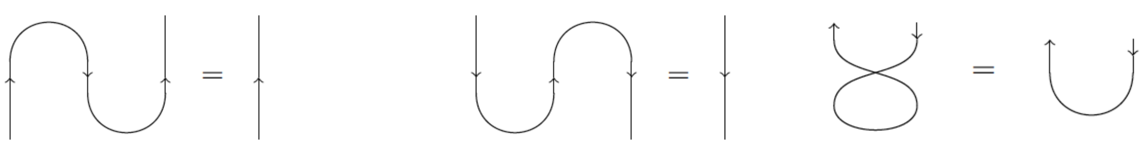
Note that from this more general perspective the typing problem with transposition of nested caps/cups vanishes by making

$$(\mathsf{A} \otimes \mathsf{B})^* \;=\; \mathsf{B}^* \otimes \mathsf{A}^*$$

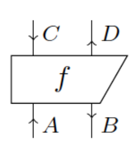
However, the analogy with wires becomes less obvious. The problem is (graphically) overcome through the introduction of a *direction* to the wires:

Thus, caps and cups are once again represented by wires, but directed wires:

And their axioms becomes



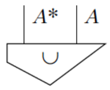
A process $\mathsf{f} : \mathsf{A} \otimes \mathsf{B}^* \longrightarrow \mathsf{C}^* \otimes \mathsf{D}$ is depicted as

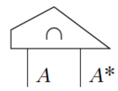


A *directed string diagram* allows any connection between two wires provided that both types and directions are compatible: types must coincide when connecting an input to an output, but should be dual when connecting ports of different polarity.

$\boxed{\text{Example: The theory of linear maps}}$

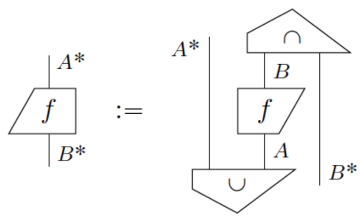
For each finite-dimensional vector space A, its dual A^* is the vector space of linear maps form A to $\mathbb{C}$, where sum and scalar multiplication are defined pointwise[3]. A basis for A^* is also obtained from the basis $\{u_i \mid i \in I\}$ of A as $\{\underline{u}_i \mid i \in I\}$ such that $\underline{u}_i u_j = \delta_{i,j}$. We now define a cap effect and cup state as follows:





$$\sum_i \underline{u}_i \otimes u_i \qquad\qquad t(v),\ \text{for each } (v \in A, t \in V^*)$$

Transposing a process $f : A \longrightarrow B$ with respect to these new caps and cups, yields



which corresponds to pre-composition with f, i.e.

$$f^*(t) \;=\; t \cdot f$$

Dagger compact closed categories.

A symmetric monoidal category C is *compact closed* if for each object A there is another object A^* and arrows

$$\epsilon_A : A \otimes A^* \longrightarrow I \qquad \text{and} \qquad \eta_A : I \longrightarrow A^* \otimes A$$

such that

$$(\epsilon_a \otimes id_A) \cdot (id_A \otimes \eta_A) \;=\; id_A$$
$$(id_{A^*} \otimes \epsilon_A) \cdot (\eta_A \otimes id_{A^*}) \;=\; id_{A^*}$$

A *dagger compact closed category* is a compact closed category C equipped with a *dagger functor* $\dagger : C \longrightarrow C$ such that

$$\epsilon_A^\dagger \;=\; \eta_{A^*}$$

where a *dagger functor* is defined by

$$A^\dagger \;=\; A \quad \text{and} \quad (f : A \longrightarrow B)^\dagger \;=\; f^\dagger : B \longrightarrow A$$

[3]i.e. $(t+s)(v) = t(v) + s(v)$ and $(\alpha\cdot)t(v) = \alpha t(v)$.

and, additionally, is involutive and respects the symmetric monoidal structure, i.e.

$$
\begin{aligned}
f &= (f^\dagger)^\dagger \\
(g \cdot f)^\dagger &= f^\dagger \cdot g^\dagger \\
(f \otimes g)^\dagger &= f^\dagger \otimes g^\dagger \\
\sigma_{A,B}^\dagger &= \sigma_{B,A}
\end{aligned}
$$