Lecture 10: The theory of quantum processes

Summary.
(1) From pure quantum maps to quantum maps.
(2) Causality.
(3) Non-determinism and quantum processes.

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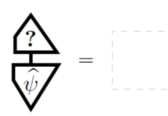
Introducing a discarding effect.

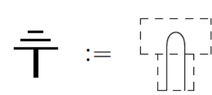
The key to provide a full picture of quantum processes is the introduction of a *discarding* effect providing a test which succeeds with certainty but does not reveal nothing about (and, of course, does not depend on) the (normalised) state that gets discarded[1].



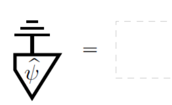
Exercise 1

Show that a discarding effect does not exist in the process theories of linear maps ou pure quantum maps.

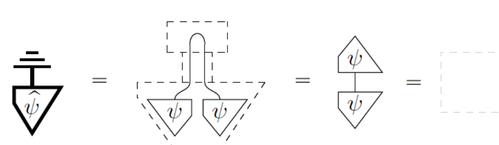
The discarding effect is defined as



which satisfies the envisaged property:



because, noting that if $\hat{\psi}$ is normalised so is ψ,



[1] The use of nnrmalised states is essencial here: suppose a discarding effect exists, thus yielding 1 when applied to an arbitrary state $\hat{\varphi}$. Clearly the application to e.g. $5\hat{\varphi}$ would return 5 rather than 1 ...

Note that this is the only possible definition. Indeed, suppose there was another effect d sending all normalised pure quantum states to 1. As discussed in the previous lecture any orthonormal basis on a type A (in the theory of linear maps) can be extended to a basis for $A \otimes A$ which is also a basis for $\hat{A}$ in the theory of pure quantum maps. Let B be corresponding normalised basis. Then applying any of the two candidates to be a discarding effect to all states in B always yields 1, thus forcing them to coincide.

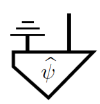
Exercise 2

Distinguish the discarding effect from cups in the theory of pure quantum maps.

Exercise 3

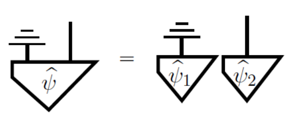
Characterise the discarding effect for types $\hat{A} \otimes \hat{B}$ and $\hat{\mathbb{C}}$. Note that $\hat{\mathbb{C}}$ is type I (the identity of $\otimes$) in the theory of pure quantum maps. Similarly, in the theory of linear maps, I is $\mathbb{C}$ (which can be regarded as the one-dimensional Hilbert space). The process we have been representing as 1, or depicting as the empty diagram, is, in any process theory, the identity on I (id_I).

Not only the discard effect is not a pure quantum effect, but, in general, reducing a pure state by discarding part of its output, i.e.

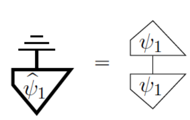


does not yield a pure quantum state. Actually, the reduced state is pure iff it is $\otimes$-separable. It is instructive to look at the proof of this claim.

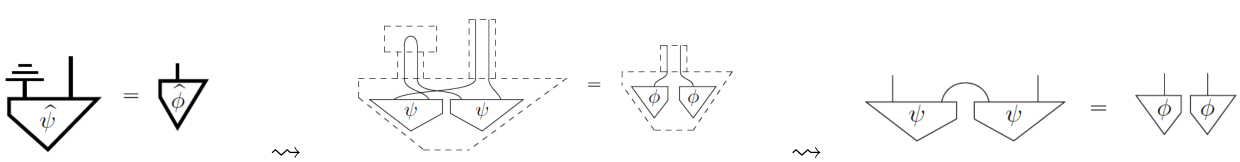
Consider, first, that the process is $\otimes$-separable, i.e.



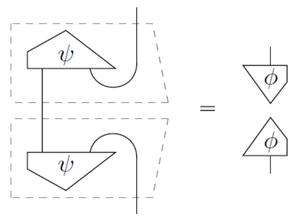
By construction,



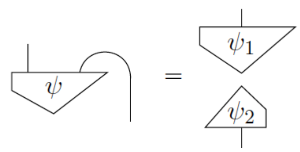
which, as a scalar in the theory of pure quantum maps, is a positive number. Thus the reduced state is a pure quantum state. For the opposite direction, assume that the reduced state is equal to a pure state $\hat{\phi}$. Unfolding,



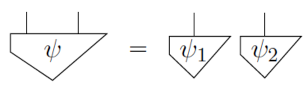
which is equivalent, by process-state duality, to



We may now resorting to a result discussed in the previous lecture stating that f is $\bullet$-separable iff $\mathsf{f}^\dagger \bullet \mathsf{f}$ is. Thus

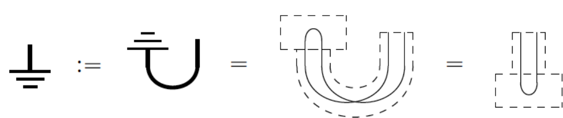


By process-state duality, ψ is ⊗-separable

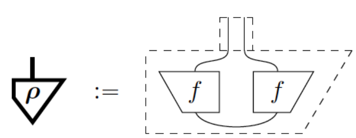


The conclusion follows from doubling this last equation.

The adjoint of the discarding effect is



In general, quantum states are obtained through the composition of pure quantum maps and discarding. Their general form is



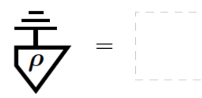
They correspond to ⊗-positive states in the theory of linear maps. Unfolding an impure quantum state and a pure one, the difference amounts to wiring, or not, the left half to the right half: so a state being *pure* or *impure* is essentially a diagrammatic notion.

Exercise 4

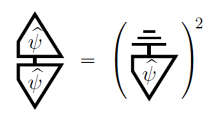
Show that, although absence of a wire illustrates purity, its presence only indicates the possibility of being impure.

Causality.

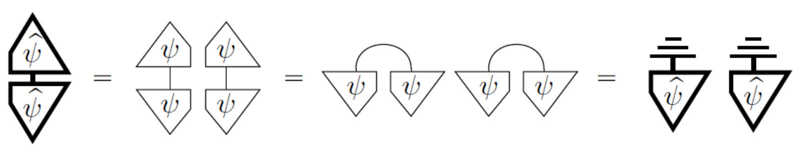
The *weight* of a quantum state ρ is the scalar resulting from its composition with discarding. Actually, it is the result of performing a trivial test on the state — testing whether it is a state. Such a test would be expected to always return 1, but such is not necessarily the case if the state results from some sort of non-determinism. Actually, states for which this scalar is 1 are the ones that occur with certainty. Formally, a. state is *causal* is this scalar is 1; in pictures



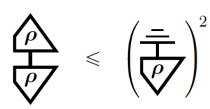
In general, a quantum state is always a combination of a causal state and the probability that it occurred. Ignoring non-determinism all states are causal. So, the causality equation basically says that *if a state is discarded, it may as well never have existed.* For (normalised) pure states (squared-)norm and weight coincide, which is a consequence of the following result: for any pure state $\widehat{\psi}$,



because

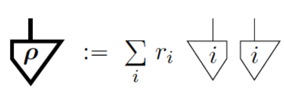


In general, however,

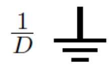


Exercise 5

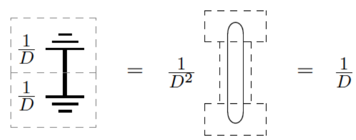
Verify this claim recalling that ρ is ⊗-positive and, therefore, by the spectral theorem, there exists a orthonormal basis and positive scalars such that



This result indicates that as a causal state becomes more impure, the (squared-)norm will go lower and lower. The limit is the completely impure state, also called the *maximally* *mixed* state which stands for a complete lack of knowledge about the system's actual state:

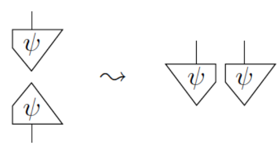


Clearly

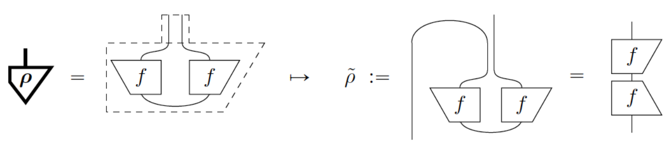


Remark

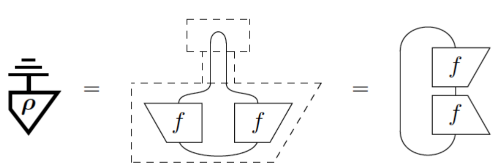
The *doubling* procedure discussed in the previous lecture is closely related to the notion of a density operator $\tilde{\psi} = |\psi\rangle\langle\psi|$, which often in textbooks replaces vectors $|\psi\rangle$ as the standard notion of a pure quantum state. Actually, the density operator has the same data as a double state: one is obtained from the other by transposing the effect ψ into the conjugate of state ψ:



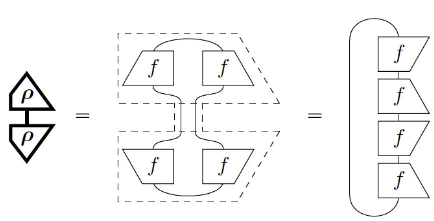
This density operator is a projector. Similarly, the density operator associated to a causal mixed state is a positive map with trace 1. Indeed, this representation is given by process-state duality



Thus, discarding a state means taking its trace



and, similarly,

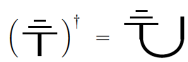


Therefore, the previous inequality becomes, in the density operator language, the well-known

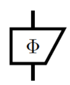
$$\mathrm{tr}(\tilde{\rho}^2) \;\leq\; \mathrm{tr}(\tilde{\rho})^2$$

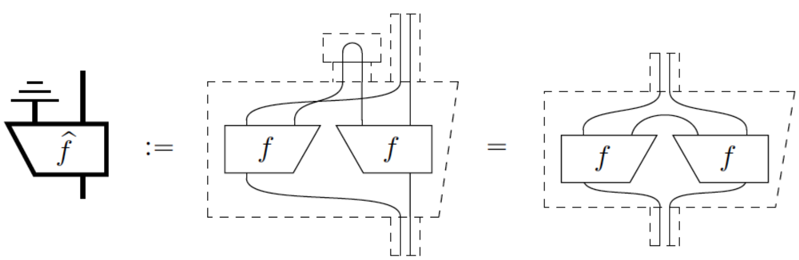
Quantum maps.

The theory of quantum maps is obtained from that of pure quantum maps by adding *discarding*. Clearly, this new theory admits string diagrams. It inherits from the pure case caps and cups, so it remains to show the existence of adjoints. The adjoint of a pure map is also a pure map, and the adjoint of discarding is

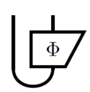


which composes a cup with discarding, making again a quantum map. Since adjoints need to preserve diagrams and all diagrams in quantum maps are made up of pure quantum maps and discarding, every quantum map has an adjoint.

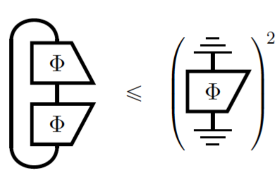
This is the general form of a quantum map 



which means that quantum maps correspond to those linear maps f which are $\otimes$-positive. The pure quantum map $\hat{\mathsf{f}}$ above is known as the *purification* of the quantum map.

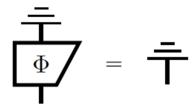
The result verified in Exercise 5 applied to a state 

provides a way to check if a quantum map is pure. Actually, it leads to



with the equality holding iff ϕ is pure.

Causality also applies to maps: A quantum map is causal if

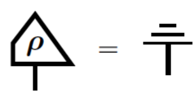


which may be read as *if the output of a process is discarded, it may as well have never happened.*

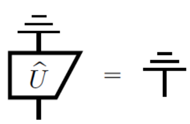
Exercise 6

Show that a causal map preserves causal states.

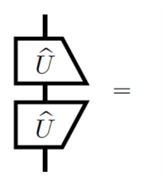
Causal effects are scarce: as shown in Exercise 3, discarding for type I amounts at doing nothing. Since effects have no outputs, causality reduces to



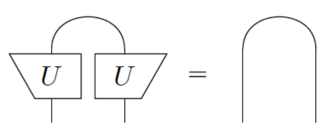
making discarding the only causal effect. Thus, there are no pure causal quantum effects. Moreover, any pure quantum map from a type $\hat{\mathrm{A}}$ to itself is causal iff it is unitary. The reason is that any isometry from $\hat{\mathrm{A}}$ to $\hat{\mathrm{A}}$ is unitary (a consequence of the dimension theorem), and one can prove that $\hat{\mathrm{U}}$ is causal iff it is an isometry; in pictures



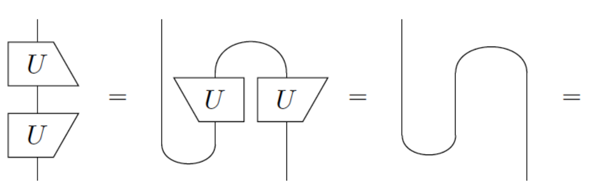
iff



Indeed, unfolding the causality equation one gets



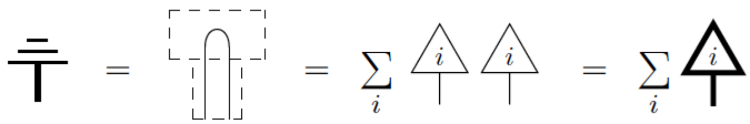
Thus,



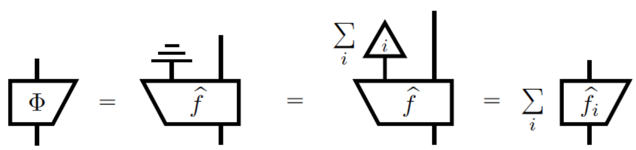
This result helps to build up intuition as causality has probably a more direct physical interpretation than being unitary.

Mixing: An alternative view.

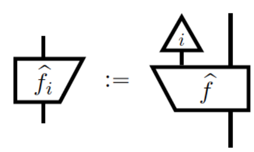
Impure quantum maps are obtained through discarding parts of a larger system. An alternative interpretation can be done as follows: First unfold the definition of the discarding effect and re-write the cap using explicit sums:



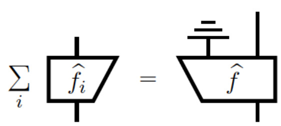
Then, any quantum map can be written as a sum of pure quantum maps



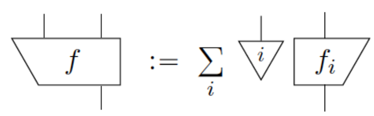
where



Conversely, any finite set of pure quantum maps is a quantum map



where



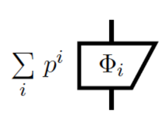
Exercise 7

Prove that the theory of quantum maps is closed for sums (which is not the case for pure quantum maps).

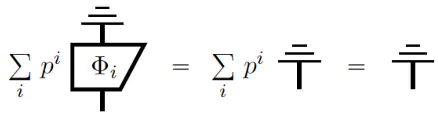
Exercise 8

Show that the sum of causal quantum maps is not necessarily causal.

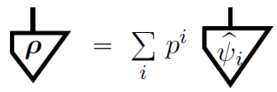
If one takes *mixtures* (i.e. *convex combinations*) instead of ordinary sums, causality is preserved. Indeed, a mixture of a family of causal quantum maps is a sum of the form



where $\sum_I p^i = 1$. Thus,



In a mixture each p^i can be interpreted as the probability of the corresponding process to happen. For example, the causal state

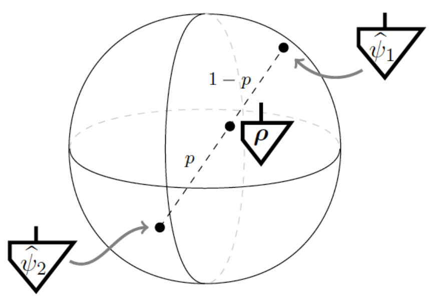


can be regarded as a system which is in one of the pure states $\hat{\psi}_i$ with probability p^i. Actually, every causal quantum state can be expressed as a mixture of pure quantum states.

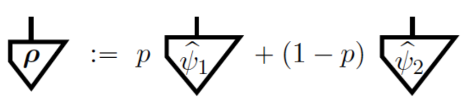
Exercise 9

Show that not every causal map can be expressed as a sum of pure causal maps.

Geometrically, if pure causal states can be represented as points on the surface of the Bloch sphere, a mixed state appears as a point inside the corresponding ball:

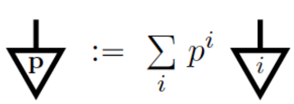


where



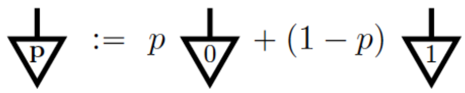
Exercise 10

Show that, given a fixed orthonormal basis, probability distributions can be represented as causal quantum states of the form

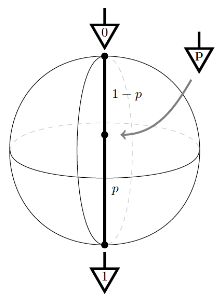


The pure states correspond to point distributions.

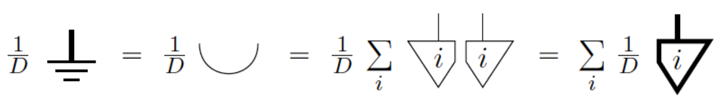
In the two-dimensional case probability distributions become



which are depicted as points in a line connecting two doubled basis states, each point corresponding to a number $p \in [0, 1]$.



There is, of course, a fundamental difference between probability distributions and quantum states. The former are uniquely decomposed into a probability distribution over point distributions (i.e. pure states), whereas the decomposition of the latter is, in general, not unique. A quantum state may decompose as many different mixtures of pure states. A typical example is the *maximally mixed state* introduced above which can be decomposed across any orthonormal basis:



Its name conveys the idea that it is equally distant from any pure state used in the decomposition. So, it can be thought as pure noise, as it does not have any bias towards any meaningful data, i.e. any pure state.

Quantum processes.

Our journey to formalise quantum processes started from the theory of linear maps to which some new ingredients were added along the way:

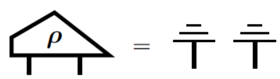
- *Doubling*, to capture probabilities as scalars, and get rid of global phases, leading to a theory of pure quantum maps.
- *Discarding*, to be able to ignore part of a system, thus capturing our lack of knowledge about its state. Such (impure) quantum maps can alternatively be described as probabilistic mixtures.

Quantum theory, however, deals with states and processes which are *non deterministic* in a fundamental sense: such non-determinism cannot be accounted for solely based on lack of knowledge about the system at hands. Regardless of how perfect is the current knowledge of the system, non-deterministic processes will not have a fixed outcome until they occur[2].

On the other hand, quantum processes are supposed to be *causal*, which put the theory out of conflict with other physical theories, namely special relativity by forbidding *faster-than-light* signalling.

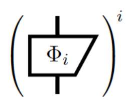
Exercise 11

Having proved in Exercise 3 that discarding a system of type $A \otimes B$ is the same as discarding individually subsystems A and B, and recalling that the only causal quantum effect is discarding itself, it is easy to conclude that all causal quantum effects are separable, i.e.

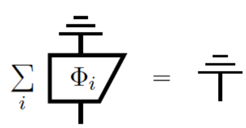


Use this fact to show that the theory of causal quantum maps does not admit string diagrams.

This discussion motivates a more general definition: A *quantum process* is a collection of quantum maps

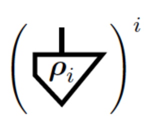


each of which called a *branch* which together satisfy the following *causality* postulate:



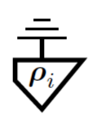
A process is deterministic if this collection is singular. When acted by a quantum process one of the branches actually occurs and constitutes the outcome of the process.

A quantum process



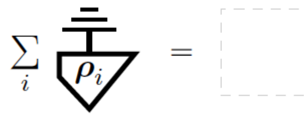
is a state in the theory. Its weight corresponds to its probability

$P(\boldsymbol{\rho}_i)$:=

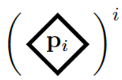


[2] cf, Einstein's famous aphorism expressing his skepticism wrt quantum mechanics — *God does not play dice.*

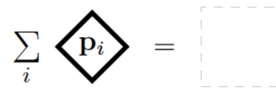
The causality requirement means that



Similarly, scalars in the theory of quantum processes are collections

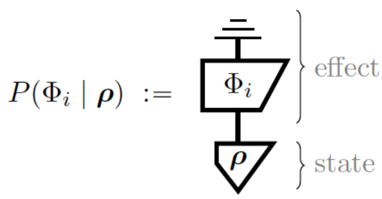


such that

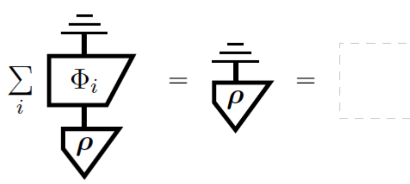


each of them thus forming a probability distribution.

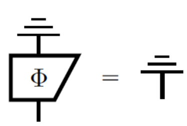
Note that it is not possible to associate a fixed probability distribution to a general quantum process. Indeed, the probabilities will depend on the state to which the process is applied. Once applied, however, probabilities can be assigned and are, as usual, computed by the Born rule



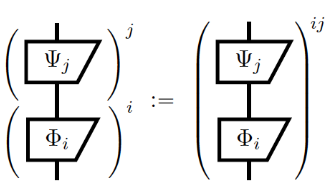
and satisfy



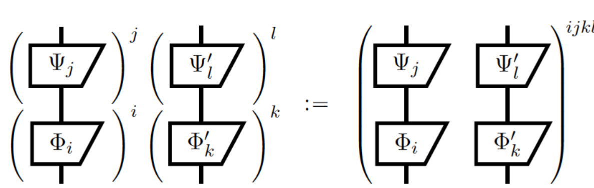
as enforced by causality. For a deterministic process this last equation boils down to the definition of a causal quantum map:



Sequential and parallel composition of quantum processes is defined to guarantee that any combination of valid branches can happen. Thus



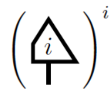
and



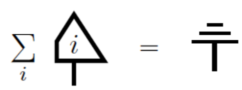
Exercise 12

Prove that causality is preserved by both sequential and parallel composition of quantum processes, which, therefore, can be organised into circuits.

Note that quantum processes admit string diagrams (once some extra notation replaces the family indexes in the pictures ...). Moreover, any quantum map can be realiazed as a quantum process. First notice that the collection of doubled effects of an orthonormal basis forms a quantum process

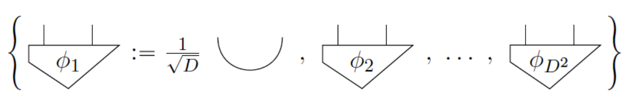


It is enough to reflect vertically the decomposition of the maximally mixed state presented above; removing the $\frac{1}{\mathsf{D}}$ yields

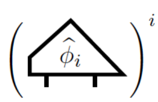


which is causal and corresponds to an orthonormal basis measurement.

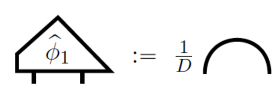
This result can be used to realise Bell effects, in an arbitrary dimension D, non-deterministically. Actually, since any normalised state can be regarded as part of an orthonormal basis, there exists one such basis including the normalised cup:



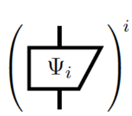
Therefore, there exists a quantum process



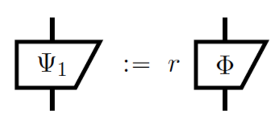
such that



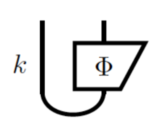
One can put together these results to prove that every quantum map can be realised non-deterministically, up to a scalar, i.e., there exists a quantum process



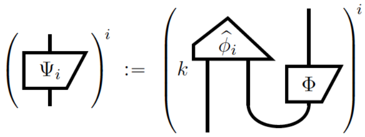
such that



for $\mathsf{r} > 0$. To verify this claim recall first that a causal state is one which occurs with certainty, thus a quantum state can always be described as a causal one tensored by a scalar. Similarly, one may chose k to make



causal. Combining with the process just built, yields



where