Towards a quantum probabilistic logic

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Overview

Quantum logic

- Quantum logic was firstly published in 1936 [Birkhoff and Von Neumann, 1936]
- Objects are results of experiments in quantum physics
  - p = "the particle has momentum in the interval [0, +1/6]"
  - q = "the particle is in the interval [1, 1]"
  - r = "the particle is in the interval [1, 3]"
  - p and (q or r) = true
  - (p and q) or (p and r) = false
- Propositions p correspond to subsets of the phase space
  - In the case of quantum physics the phase space is the Hilbert space, and the possible results correspond to Closed linear subspaces, as observables correspond to Hermitic operators $(O = O^\dagger)$

Quantum logic

- Measurements are projections
- A logic can be built upon algebraic operations over such spaces (unions, intersections, complements)

$$\varphi ::= p \mid \perp \mid \sim \varphi \mid \varphi \wedge \varphi \mid \varphi \sqcup \varphi \tag{1}$$

- The main issue is *Complentarity*: the Measurement of the system changes the system state
    - The lattice of propositions is not distributive, hence no natural notion of implication

Quantum programs

What are quantum programs ?

$$p : H \to H \tag{2}$$

Restrictions:

- Transitions are unitary $U.U^{\dagger} = I$
- Measurements spoil the system state $O^{\dagger} = O$
- No cloning theorem - Quantum Information cannot be copied! $\cancel{x = y}$, $\cancel{x = x + 1}$

Advantages:

- Quantum parallelism!
  - $O\left|\Psi\right\rangle = O\sum_n \left|\Psi_n\right\rangle = O\left|\Psi_1\right\rangle + O\left|\Psi_2\right\rangle + \ldots + O\left|\Psi_n\right\rangle$

Quantum Dynamic logic

Dynamic logic: "The logic of Quantum Programs" Baltag & Smets [Baltag and Smets, 2004, Baltag et al., 2014]
Syntax:

$$\varphi ::= p|\varphi \vee \varphi|\neg\varphi|[\pi]\varphi|K_I\varphi|P^{\geq r}\varphi \tag{3}$$

$$\pi ::= unitary|\varphi?|\pi;\pi|\pi \cup \pi \tag{4}$$

(unitary includes Hadamard, CNOT, X, Y, Z, Rotational Gates, Toffoli gates)

Quantum Dynamic logic

Modalities:

- $[\pi]\varphi$ - φ holds on the successful execution of program π
- $K_I\varphi$ - φ holds in a sub-system of the current quantum state. This is what makes this logic an epistemic logic.
- $P^{\geq r}\varphi$ - The condition φ holds with a probability greater than r. This is what makes the logic a probabilistic one.

$\perp ::= \varphi \wedge \neg\varphi$	$< \pi > \varphi := \neg[\pi]\neg\varphi$	$A\varphi := \neg E \neg\varphi$
$\top := \neg\perp$	$\diamond\varphi :=< \varphi? > \top$	$p^{<R}\varphi := \neg p^{\geq r}\varphi$
$\sim \varphi := [\varphi?]\perp$	$:= \neg \diamond \neg\varphi$	$p^{>r}\varphi := \neg p^{\leq R}\varphi$
$\varphi \sqcup \psi :=\sim (\sim \varphi \wedge \sim \psi)$	$E\varphi := \diamond\diamond\varphi$	$p\varphi$
$\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$	$p^{\leq r}\varphi := p^{>(1-r)} \sim \varphi$	
$[\phi?^{\geq r}]\psi := p^{\geq r}\varphi \to [\varphi?]\psi$	$\varphi \to \psi := \neg(\varphi \wedge \neg\psi)$	

Quantum Dynamic logic

Semantics:

- Modalities
  - $[\![p]\!]$ - $[\![p]\!] \subseteq \Sigma$, corresponds to a closed linear subspace of H
  - $[\![\varphi \vee \Psi]\!]$ - $s \in [\![\varphi \vee \Psi]\!]$ iff either $s \in [\![\varphi]\!]$, or $s \in [\![\Psi]\!]$ so that $[\![\varphi \vee \Psi]\!] = [\![\varphi]\!] \cup [\![\Psi]\!]$
  - $[\![\neg\varphi]\!]$ - $s \in [\![\neg\varphi_q]\!]$ iff $s \notin [\![\varphi_q]\!]$, so that $[\![\neg\varphi_q]\!] = \Sigma \backslash [\![\varphi_q]\!]$, the Boolean complement of $[\![\varphi_q]\!]$
  - $[\![[\varphi?]\Psi]\!]$ - $\{s | Proj_{[\![\phi]\!]}(v) \in [\![\bar{\Psi}]\!]$ for all $v \in s\} = \widetilde{Proj_{[\![\varphi]\!]}}[[\![\bar{\varphi}]\!]]$
  - $[\![[\varphi?]\Psi]\!]$ - $\neg[\![\varphi]\!] \sqcup ([\![\varphi]\!] \cap [\![\Psi]\!])$

Quantum Dynamic logic

- $[\![K_I\varphi]\!] \Longleftrightarrow t \in [\![\varphi]\!]$, for every $t \sim_I s \Longleftrightarrow U(s) \in [\![\varphi]\!]$ for every I-Remote U
  - Indistinguishability:

$$s{\sim}_I t \Longleftrightarrow s_I = t_I \Longleftrightarrow tr_{N/I}(p_v) = tr_{N/I}(p_w) \qquad (5)$$

    for unitary $w, v \in t$
  - Indistinguishability (I-Remote):

$$s{\sim}_I t \Longleftrightarrow t = U(s) \qquad (6)$$

    for some I-remote U $(U : H \to H, U = Id_I \otimes U_{N/I})$
- $[\![P^r\varphi]\!]$ - $\langle v|Proj_{[\![\varphi]\!]}|v\rangle \geq r$ for all unit vectors $v \in s$

Quantum Dynamic logic

Quantum programs: $[\![u]\!]$ - $[\![u]\!] : \Sigma \to \Sigma$, corresponds to a unitary transformation in H

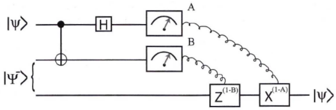
- Dynamic Modalities
  - $[\![[\pi_q]\varphi_q]\!]$ - $s \in [\![[\pi_q]\varphi_q]\!] \iff t \in [\![\varphi]\!]$ whenever $s \xrightarrow{\pi} t$
  - $s \xrightarrow{[\![u]\!]} t = \{s | [\![u]\!](s) \in [\![\varphi]\!]\} = [\![u]\!]^{-1}[\![\varphi]\!]$
  - $s \xrightarrow{[\![\varphi_q?]\!]} t = \{s | Proj_{[\![\varphi]\!]}(v) \in [\![\Psi]\!]$ for all $v \in s\}$
  - $s \xrightarrow{[\![\pi_q;\pi_q]\varphi_q]\!]} t = [\![[\pi_1][\pi_2]\varphi]\!]$
  - $s \xrightarrow{[\![\pi_q \cup \pi_q]\!]} t = [\![[\pi_1]\varphi \wedge [\pi_2]\varphi]\!]$

Quantum Dynamic logic

Hilbert Calculus

- Hilbert calculus is different between the two logics [Baltag and Smets, 2004], [Baltag et al., 2014]. Both are normal
- [Baltag and Smets, 2004] has characteristic axioms for quantum gates, entanglement, general quantum axioms

Proof of teleportation protocol



$$\pi = \bigcup_{x,y \in \{0,1\}} CNOT_{1,2}; H_1; (x_1 \wedge y_2)?; X_3^y; Z_3^x \tag{7}$$

$$\vdash \varphi_1 \wedge \beta_{00}^{2,3} \rightarrow [\pi]\varphi_3 \tag{8}$$

Quantum Dynamic logic

Proof:

$$
\begin{array}{l}
\vdash \varphi_1 \to \varphi_1 \\
\hline
\vdash \varphi_1 \to [Z_1^x; X_1^y; X_1^y; Z_1^x]\varphi_1 \\
\hline
\vdash \varphi_1 \wedge id_{2,3} \to [(Z_1^x; X_1^y)_{1,2}?]; [X_3^y; Z_3^x]\varphi_3 \\
\hline
\vdash \varphi_1 \wedge \beta_{00}^{2,3} \to \beta_{1,2}^{x,y}; [X_3^y; Z_3^x]\varphi_3 \\
\hline
\vdash \varphi_1 \wedge \beta_{00}^{2,3} \to [\bigcup_{x,y\in\{0,1\}} CNOT_{1,2}; H_1; (x_1 \wedge y_2)?; X_3^y; Z_3^x]\varphi_3
\end{array}
$$

Proof of Grover algorithm [Baltag et al., 2014]

$$
\begin{aligned}
QSA := {} & Ora(O) \wedge CState(p) \wedge A\left(p \wedge 0_n \to [O]1_n\right) \wedge A\left(p \wedge 1_n \to [O]0_n\right) \wedge \underline{\mathbf{0}} \wedge 1_n \\
& \to [H_0; \cdots; H_n][O; H_0; \cdots; H_{n-1}; P; H_0; \cdots; H_{n-1}]^k P^{>0.5} p.
\end{aligned}
$$

The quantum probabilistic logic

Objective:

Design a logic that can reason about actual quantum-probabilistic programs in practice

Motivation:

Quantum algorithms may require classical post-processing. Example: Shor algorithm

State of the art:

Several logics for quantum programs: Adams [Adams et al., 2014], Ying [Ying, 2011], Mateus [Mateus and Sernadas,]
Baltag & Smets [Baltag et al., 2014]

The quantum probabilistic logic

Phases are an issue?

- The phase is necessary to reason about the Shor algorithm: Rotational gates
- Global and local phases

$$|\Psi\rangle = -|\Psi\rangle\,; |0\rangle + |1\rangle \neq |0\rangle - |1\rangle\,; \tag{9}$$

- Phase arithmetics

$$e^{i\theta_1} * e^{i\theta_2} = e^{i\theta_1 + \theta_2} \tag{10}$$

- Deal with limits (wishfull thinking) $\sum_0^\infty e^{i\theta} = e^{i\phi}$

A possible solution is to adopt a simpler logic targeted to deal with the known structures in quantum algorithms!

The quantum probabilistic logic ii

A dynamic logic for quantum-probabilistic programs (combine with [Kozen, 1981]):
A programming language:

$$term ::= const | var | var + var | var * var | var - var | var / var$$

$$\pi_p ::= x := term | x := rnd | x := meas(\pi_q) | (\varphi_p?) | \pi_p ; \pi_p | a\pi_p + b\pi_p | \pi_p *$$

$$\pi_q ::= unitary | \varphi_q ? | \pi_q ; \pi_q | \pi_q \cup \pi_q$$

The quantum probabilistic logic ii

Modalities

$$\varphi_p ::= f|\varphi_p + \varphi_p|r\varphi_p| < \pi_p > \varphi_p|\varphi_p.\varphi_p \tag{11}$$

$$\phi_p = \varphi_p <= \varphi_p|\varphi_p \to \varphi_p \tag{12}$$

$$\varphi_q ::= p_q|\varphi_q \vee \varphi_q|\neg\varphi_q|[\pi]\varphi_q|K_I\varphi_q|P^{\geq r}\varphi_q \tag{13}$$

Interactions between modalities?

The quantum probabilistic logic ii

Logic is hierarchic?
What if the language was something as follows:

$$\pi ::= g_q | (x := term)_p | (x := rnd)_p | (\varphi?)_{p,q} | (\pi)_{p,q}; (\pi)_{p,q} | \pi_{q,p} \cup \pi_{q,p} \tag{14}$$

i.e. if both processes could interact with each other? Could we model noise with this?

Conclusions

Potential applications

- Quantum algorithms always involve interaction with the classical world. Ex: Shor algorithm
- A quantum-probabilistic logic may be used in quantum processes that involve noise

Future interesting lines of work:

- Find software tools to make this logic applicable in practice
- Use Higher-order categories in the probabilistic side. Inspiration: [Heunen et al., 2017]
- Use quantitative reasoning in dealing with language developed in this work. Inspiration: [Mardare et al., 2016]

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Questions ?